ON THE CAUCHY PROBLEM FOR FRIEDRICHS SYSTEMS

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DFG Deutsche Forschungsgemeinschaft German Research Foundation

EINSTEIN 1915

Classical spacetime $\longleftrightarrow$ Lorentzian manifold

- *globally hyperbolic spacetime*: $M = \mathbb{R} \times \Sigma$ with metric $g = -\beta^2 dt^2 + h_t$
- classical fields: section of a vector bundle $E \to M$
- dynamics: (pseudo)-differential (hyperbolic) operator $P : \Gamma(E) \to \Gamma(E)$

When the Cauchy problem for P is well-posed?

$\partial\Sigma = \emptyset$: CLASSICAL RESULTS --- (Hadamard; Leray; Hörmander; Friederich, ...)

$\partial\Sigma \neq \emptyset$: DEPENDS ON THE BOUNDARY CONDITIONS

GOAL: **study the Cauchy problem for Friedrichs systems**

... and on the menù of the day we have:

Wave equation - -- Dirac equation - -- Klein-Gordon equation

but also Diffusion-Reaction equations (e.g. the Heat equation)

Outline of the Talk

- **Geometric Preliminaries**
- **Friedrichs Systems of constant characteristic**
- **Admissible Boundary conditions**
- **Energy estimates**
- **Existence and uniqueness of strong solutions**
- **Differentiability of the solutions**
- **Outlook**

Based on

S. M. and N. Ginoux *"On the Cauchy problem for Friedrichs systems on globally hyperbolic manifolds with timelike boundary"* arXiv:2007.02544

Geometric preliminaries

- M is connected, time-oriented, oriented smooth manifold with smooth boundary ∂M
- g is Lorentzian metric and ∂M is timelike

Few important definitions

- *Temporal function:* $t \in C^\infty(M, \mathbb{R})$ strictly increasing on future directed causal curve and ∇t is timelike everywhere and past-pointing
- *Cauchy hypersurface* Σ: if each inextendible timelike curve $\gamma \cap \Sigma = \{\text{pt}\}$
- *Globally hyperbolic*: M *strongly causal* and $\forall p, q \in M$, $J^+(p) \cap J^-(q)$ compact

Bernal and Sánchez (2005) – Aké, Flores and Sánchez (2019):

M is globally hyperbolic (with timelike boundary)

$$\Updownarrow$$

Exists a Cauchy temporal function ($t^{-1}(s) := \Sigma_s$ is a Cauchy) and $\nabla t \in T\partial M$

$$\Downarrow$$

M isometric to $\mathbb{R} \times \Sigma$ with metric $-\beta^2 dt^2 + h_t$, where $\beta \in C^\infty(M, (0, \infty))$

Friedrichs systems of constant characteristic

- $E \to M$ be a $\mathbb{K}$-vector bundle with finite rank N and sesquilinear fiber metric $\prec \cdot | \cdot \succ$

Definition: a 1^{st} order S is called *symmetric system* if

(S) $\sigma_S(\xi)\colon E_p \to E_p$ is Hermitian with respect to $\prec \cdot | \cdot \succ$, $\forall \xi \in T_p^*M$ and $\forall p \in M$.

Additionally, we say that S is *hyperbolic* respectively *positive* if it holds:

(H) $\prec \sigma_S(\tau) \cdot | \cdot \succ$ is positive definite on E_p, for any future-directed timelike $\tau \in T_p^*M$

(P) $\phi \mapsto \prec \Re e(\mathsf{S}^\dagger + \mathsf{S})\phi \,|\, \phi \succ_{\Sigma_t} \geq c_t \prec \phi \,|\, \phi \succ_{\Sigma_t} \in C^0(\mathbb{R})$ with $c_t > 0$.

We call **Friedrichs system**, any *symmetric* system S which is *hyperbolic* or *positive*.

If $\dim\ker \sigma_S(\mathtt{n}^\flat)$ is constant then S is **of constant characteristic** (where $\mathtt{n} \perp \partial M$).

Example: $E = \mathbb{C}^N \times [0,\infty) \times \mathbb{R}^n \to ([0,\infty) \times \mathbb{R}^n, \eta)$ with $\prec \,|\, \succ = \langle \,|\, \rangle_{\mathbb{C}^N}$

$$\mathsf{S} := A_0(p)\partial_t + \sum_{j=i}^{n} A_j(p)\partial_{x_j} + B(p)$$

(S) $A_0 = A_0^\dagger\,,\ A_j = A_j^\dagger$ $\qquad$ (H) $\sigma_S(dt + \sum_j \alpha_j dx_j) = A_0 + \sum_{j=1} \alpha_j A_j > 0\,.$

(P) $\Re e(B + B^\dagger - \dfrac{\partial_t(\sqrt{g}A_0)}{\sqrt{g}} - \sum_{j=1}^{n} \dfrac{\partial_{x_j}(\sqrt{g}A_j)}{\sqrt{g}}) > 0\,,$ $(C.C.)$ $\dim\ker A_z$ is const.

Basic Properties

Lemma: If $\prec \cdot | \cdot \succ$ is indefinite and S is a symmetric hyperbolic system

(I) $\langle \cdot | \cdot \rangle := \prec \sigma_S(dt) \cdot \, | \cdot \succ$ is positive Hermitian metric;

(II) $\mathfrak{S} = \sigma_S(dt)^{-1}\mathsf{S}$. is symmetric hyperbolic system

(III) Cauchy problem $(\mathfrak{S}) \Longleftrightarrow$ Cauchy problem (S) set: $\Big(\sigma_S(dt)^{-1}\mathfrak{f}, \mathfrak{h}) \leftarrow (\mathfrak{f}, \mathfrak{h})\Big)$

Lemma: If S is a symmetric hyperbolic system in $M_T := t^{-1}(t_0, t_1)$

(I) $\mathfrak{S} := \mathsf{S} + \lambda\sigma_S(dt)$ is symmetric hyperbolic system

$$\begin{cases} \mathfrak{S}\widetilde{\Psi} = \widetilde{\mathfrak{f}} \\ \widetilde{\Psi}|_{\Sigma_0} = \widetilde{\mathfrak{h}} \\ \widetilde{\Psi} \in \mathsf{B} \end{cases} \Longleftrightarrow \begin{cases} \mathsf{S}\Psi = \mathfrak{f} \\ \Psi|_{\Sigma_0} = \mathfrak{h} \\ \Psi \in \mathsf{B}, \end{cases}$$

where $\widetilde{\mathfrak{f}} = e^{-\lambda t}\mathfrak{f}$, $\widetilde{\mathfrak{h}} = \mathfrak{h}$ and $\widetilde{\Psi} = e^{-\lambda t}\Psi$.

(II) $\forall\ U \subset M$ compact, $\exists\ \lambda \equiv \lambda(U)$ such that $\mathfrak{S}$ is a positive symmetric system

Lemma: If S is a symmetric system, we have a Green Formula

(Green Formula) $(\mathsf{S}\Phi \,|\, \Phi)_M - (\Phi \,|\, \mathsf{S}^\dagger\Phi)_M = (\Phi \,|\, \sigma_S(\mathtt{n}^\flat)\Phi)_{\partial M}$

Example I: The classical Dirac operator

- $M = (M, g)$ is a globally hyperbolic spin manifold with timelike boundary;
- $\mathbb{S}M$ is a *spinor bundle*: $\mathbb{C}$-vector bundle with indefinite sesquilinear metric

$$\prec \cdot | \cdot \succ : \mathbb{S}_p M \times \mathbb{S}_p M \to \mathbb{C}$$

and a Clifford multiplication, i.e. fiber-preserving map $\gamma \colon TM \to \mathrm{End}(\mathbb{S}M)$

Dirac operator: $\mathsf{D} := \gamma \circ \nabla^{\mathbb{S}} \colon \Gamma(\mathbb{S}M) \to \Gamma(\mathbb{S}M)$ which in local coordinates reads

$$\mathsf{D} = \sum_{\mu=0}^{n} \varepsilon_\mu \gamma(e_\mu) \nabla^{\mathbb{S}}_{e_\mu}$$

- $(e_\mu)_{\mu=0,\dots,n}$ is a local orthonormal Lorentzian frame of TM and $\varepsilon_\mu := g(e_\mu, e_\mu)$
- $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u, v)$ for every $u, v \in T_p M$ and $p \in M$.

Remarks:

(i) Topological obstruction to existence of a spinor bundle;

(ii) Existence of spinor bundles on parallelizable manifolds;

(iii) D is nowhere characteristic.

Example II: The geometric wave operator

-M is a globally hyperbolic with timelike boundary and $g = -\beta^2 dt^2 + h_t$;

- V be an Hermitian vector bundle of finite rank;

- P is a normally hyperbolic operator i.e. $P = -\mathrm{tr}(\nabla\nabla) + c$ and principal symbol

$$\sigma_P(\xi) = -g(\xi^\sharp, \xi^\sharp) \cdot \mathrm{Id}_V \,, \qquad \text{for every } \xi \in T^*M.$$

A normally hyperbolic operator P can be rewritten as SYMM. HYPERBOLIC SYST.

$$\mathsf{S} := (A_0 \nabla_{\partial_t} + A_\Sigma \nabla^\Sigma + C)$$

$$\Psi := \begin{pmatrix} \nabla_{\partial_t} u \\ \nabla^\Sigma u \\ u \end{pmatrix} \quad A_0 := \begin{pmatrix} \frac{1}{\beta^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_\Sigma = \begin{pmatrix} 0 & -\mathrm{tr}_{h_t} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} \text{suitable} \end{pmatrix}$$

Remarks:

(i) Cauchy problem $(P) \Longleftrightarrow$ Cauchy problem (S);

(ii) S is of constant characteristic:

$$\sigma_S(\mathtt{n}^\flat) = \begin{pmatrix} 0 & -\mathtt{n}^\flat \lrcorner & 0 \\ -\mathtt{n}^\flat \otimes & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example III and IV: The Klein-Gordon operator and the Heat operator

-$M = (M, g)$ is a globally hyperbolic spin manifold with timelike boundary;

- P is the Klein-Gordon operator i.e. $P = -\mathrm{tr}(\nabla\nabla) + m^2$

The Klein-Gordon operator P can be rewritten as SYMM. POSITIVE SYST.

$$\mathrm{S} = \begin{pmatrix} 0 & -\mathrm{tr} \\ -1 & 0 \end{pmatrix} \nabla + \begin{pmatrix} m^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with} \qquad \Psi = \begin{pmatrix} u \\ \nabla u \end{pmatrix}$$

- H is the Forward Heat operator i.e. $\mathrm{H} = \nabla_{\partial_t} - \mathrm{tr}(\nabla^\Sigma \nabla^\Sigma)$

- H_b is the Backward Heat operator i.e. $\mathrm{H} = \nabla_{\partial_t} + \mathrm{tr}(\nabla^\Sigma \nabla^\Sigma)$

The H and H_b can be rewritten as a SYMM. POSITIVE SYST.

$$\widetilde{\mathrm{S}}_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nabla_{\partial_t} + \begin{pmatrix} 0 & \pm\mathrm{tr} \\ -1 & 0 \end{pmatrix} \nabla^\Sigma + \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with} \qquad \Psi = \begin{pmatrix} u \\ \nabla^\Sigma u \end{pmatrix}$$

Remarks:

(i) Cauchy pr. (P) $\iff$ Cauchy pr. (S) and Cauchy pr. (H) $\iff$ Cauchy pr. $(\widetilde{\mathrm{S}}_\lambda)$

(ii) S and $\widetilde{\mathrm{S}}_\lambda$ are of nowhere characteristic

$$\sigma_S(\mathtt{n}^\flat) = \begin{pmatrix} 0 & -\mathtt{n}^\flat \lrcorner \\ -\mathtt{n}^\flat \otimes & 0 \end{pmatrix}$$

Admissible boundary conditions

Definition [Friedrichs]: A boundary space B_{adm} for S is called **admissible** if

- The quadratic form $\Psi \mapsto \prec \sigma_S(\mathtt{n}^\flat)\Psi \,|\, \Psi \succ$ is positive semi-definite on B_{adm};
- $\operatorname{rank} \mathsf{B}_{adm} = \#$ pointwise non-negative eigenvalues of $\sigma_S(\mathtt{n}^\flat)$ counting multiplicity.

Remarks:

(i) The adjoint boundary space is defined by $\mathsf{B}^\dagger_{adm} := \big(\sigma_\mathsf{S}(\mathtt{n}^\flat)(\mathsf{B}_{adm})\big)^\perp$, i.e.

$$\{\Phi \in \Gamma(E|_{\partial M}) \,|\, \text{for any } \Psi \in \mathsf{B}_{adm} \text{ it holds } \prec \sigma_\mathsf{S}(\mathtt{n}^\flat)\Psi \,|\, \Phi \succ = 0\}\,.$$

(ii) $\Phi \mapsto \prec \sigma_S(\mathtt{n}^\flat)\Phi \,|\, \Psi \succ$ is negative semi-definite on $\mathsf{B}^\dagger_{adm}$

Examples for classical Dirac operators:

Lorentzian MIT boundary space is the range of $\pi_{\mathrm{Lor}} := \frac{1}{2}\left(\mathrm{Id} \pm \imath\gamma(\mathtt{n})\right)$

$$\prec \sigma_\mathsf{D}(\mathtt{n}^\flat)\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ = \prec \gamma(\mathtt{n})\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ = \imath \prec \pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ = 0$$

Riemannian MIT boundary space is the range of $\pi_{\mathrm{Riem}} := \frac{1}{2}\left(\mathrm{Id} - \frac{1}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\right)$

$$\begin{aligned}\prec \sigma_\mathsf{D}(\mathtt{n}^\flat)\pi_{\mathrm{Riem}}\psi \,|\, \pi_{\mathrm{Riem}}\psi \succ &= \prec \gamma(\mathtt{n})\pi_{\mathrm{Riem}}\psi \,|\, \pi_{\mathrm{Riem}}\psi \succ \\ &= \frac{1}{\beta} \prec \gamma(\partial_t)\pi_{\mathrm{Riem}}\psi \,|\, \pi_{\mathrm{Riem}}\psi \succ \geq 0\end{aligned}$$

Admissible boundary conditions

Definition [Friedrichs]: A boundary space B_{adm} for S is called **admissible** if

- The quadratic form $\Psi \mapsto \prec \sigma_S(\mathtt{n}^\flat)\Psi \,|\, \Psi \succ$ is positive semi-definite on B_{adm};
- $\operatorname{rank} \mathsf{B}_{adm} = \#$ pointwise non-negative eigenvalues of $\sigma_S(\mathtt{n}^\flat)$ counting multiplicity.

Remarks:

(i) The adjoint boundary space is defined by $\mathsf{B}^\dagger_{adm} := \big(\sigma_\mathsf{S}(\mathtt{n}^\flat)(\mathsf{B}_{adm})\big)^\perp$, i.e.

$$\{\Phi \in \Gamma(E|_{\partial M}) \,|\, \text{for any } \Psi \in \mathsf{B}_{adm} \text{ it holds } \prec \sigma_\mathsf{S}(\mathtt{n}^\flat)\Psi \,|\, \Phi \succ = 0\}\,.$$

(ii) $\Phi \mapsto \prec \sigma_S(\mathtt{n}^\flat)\Phi \,|\, \Psi \succ$ is negative semi-definite on $\mathsf{B}^\dagger_{adm}$

Examples for geometric wave operator:

Neumann like-boundary condition: $\nabla^\Sigma_\mathtt{n} u = 0 \implies \mathsf{B}^{N.}_{adm} = \ker \begin{pmatrix} 0 & \mathtt{n}\lrcorner & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Transparent boundary condition: $\nabla^\Sigma_\mathtt{n} u = -b\nabla_{\partial_t} u \implies \mathsf{B}^{T.}_{adm} = \ker \begin{pmatrix} b & \mathtt{n}\lrcorner & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Examples for the Klein-Gordon operator and Heat operator:

Robin like-boundary condition: $a\nabla_\mathtt{n} u = bu \implies \mathsf{B}^{N.}_{adm} = \ker \begin{pmatrix} -b & a\mathtt{n}\lrcorner \\ 0 & 0 \end{pmatrix}$

The Cauchy problem

Theorem: there exists a unique strong solution for the Cauchy problem if

- $M_T := t^{-}(t_a, t_b)$ is a globally hyperbolic times trip with timelike boundary
- S is a Friedrich system with admissible boundary conditions B_{adm}

Full-regularity for Friedrichs systems cannot be expected!

Backward Heat equation can be rewritten as a symmetric positive system

Theorem: The Cauchy problem is well-posed if

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary
- S is a Friedrichs system with $\prec \sigma_{\mathrm{S}}(dt) \cdot | \cdot \succ > 0$
- B_{adm} is an admissible boundary conditions
- $(\mathfrak{h}, \mathfrak{f})$ are Cauchy data satisfying compatibility condition up to any order

Energy estimates: symmetric hyperbolic systems

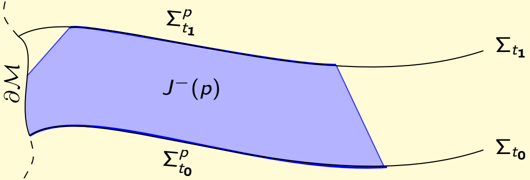
Theorem (symmetric hyperbolic system)

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary
- S is symmetric hyperbolic with admissible boundary condition B_{adm}

Then for each $t_0 \in t(M)$ $\exists$ $C > 0$ s. t. $\forall$ $t_1 \geq t_0$ it holds $\forall \Psi \in \mathsf{B}_{adm}$

$$\int_{\Sigma^p_{t_1}} |\Psi|^2 d\mu_{t_1} \leq C e^{C(t_1 - t_0)} \int_{t_0}^{t_1} \int_{\Sigma^p_s} |\mathsf{S}\Psi|^2 d\mu_s ds + e^{C(t_1 - t_0)} \int_{\Sigma_{t_0^p}} |\Psi|^2 d\mu_{t_0}$$

where $t \colon M \to \mathbb{R}$ be a Cauchy temporal function and $\Sigma^p_s := J^-(p) \cap \Sigma_s$



Energy estimates: symmetric hyperbolic systems

Theorem (symmetric hyperbolic system)

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary
- S is symmetric hyperbolic with admissible boundary condition B_{adm}

Then for each $t_0 \in t(M)$ $\exists$ $C > 0$ s. t. $\forall$ $t_1 \geq t_0$ it holds $\forall \Psi \in \mathsf{B}_{adm}$

$$\int_{\Sigma^p_{t_1}} |\Psi|^2 d\mu_{t_1} \leq Ce^{C(t_1 - t_0)} \int_{t_0}^{t_1} \int_{\Sigma^p_s} |\mathsf{S}\Psi|^2 d\mu_s ds + e^{C(t_1 - t_0)} \int_{\Sigma_{t^p_0}} |\Psi|^2 d\mu_{t_0}$$

where $t \colon M \to \mathbb{R}$ be a Cauchy temporal function and $\Sigma^p_s := J^-(p) \cap \Sigma_s$

Corollary: finite speed of propagation, *i.e.* supp $\Psi \subset \mathcal{V} := J(\text{supp } f) \cup J(\text{supp } h)$

Proposition (reduction to compact Cauchy surfaces):

- $(M, g) = (\mathbb{R} \times \Sigma_0, -\beta^2 dt^2 + h_t)$ with Σ_0 non-compact Cauchy surface
- $(\mathbb{R} \times U, -\beta^2 dt^2 + h_t)$ for any relatively compact U with smooth boundary $U \subset \Sigma_0$

If Cauchy problem for S can be solved for any $U \subset \Sigma_0$ with B_{adm} along ∂U

$$\Downarrow$$

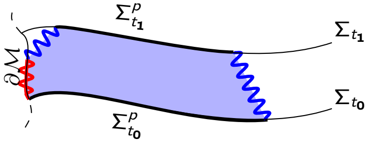
the Cauchy problem for S can be solved on M

Energy estimates: symmetric hyperbolic systems

Sketch of the proof:

-n-differential form:

$$\omega := \sum_{j=0}^{n} \Re e \left(\prec \sigma_{\mathsf{S}}(b_j^\flat)\Psi \,|\, \Psi \succ \right) b_j \lrcorner \mathrm{vol}_M$$



- Stokes' theorem for manifold with Lipschitz boundary yields

$$\int_K d\omega = \int_{\partial K} \omega = \int_{\Sigma^p_{t_1}} \omega - \int_{\Sigma^p_{t_0}} \omega + \int_{\text{red}} \omega + \int_{\text{blue}} \omega$$

- Hyperbolicity of S $\Longrightarrow \int_{\text{blue}} \omega \geq 0$ while $\Psi \in \mathsf{B}_{adm} \Longrightarrow \int_{\text{red}} \omega \geq 0$

$$\int_{\Sigma^p_{t_1}} |\Psi|^2 d\mu_1 - \int_{\Sigma^p_{t_0}} |\Psi|^2 d\mu_0 \leq \int_K d\omega \leq C \int_{t_0}^{t_1} \int_{\Sigma^p_s} (|\Psi|^2 + |\mathsf{S}\Psi|^2) d\mu_s ds$$

- By setting $h(s) := \int_{\Sigma^p_s} |\Psi|^2 d\mu_s \,, \quad \alpha(t_1) := C \int_{t_0}^{t_1} \int_{\Sigma^p_s} (|\Psi|^2 + |\mathsf{S}\Psi|^2) d\mu_s ds + \int_{\Sigma^p_0} |\Psi|^2 d\mu_0$

and using Grönwall, we obtain: $h(t_1) \leq \alpha(t_1) + C \int_{t_0}^{t_1} h(s) ds \leq \alpha(t_1) e^{C(t_1 - t_0)}$

□

Energy estimates: Friedrich systems

Remarks: If S is a symmetric hyperbolic systems on $M = \mathbb{R} \times \Sigma$, then

(i) if Σ is non-compact $\to \widetilde{\Sigma}$ compact

(ii) $\mathsf{S} \to \mathsf{S}_\lambda := \mathsf{S} + \lambda \sigma_{\mathsf{S}}(dt)$ is symmetric positive system

Theorem (Friedrich systems)

- $M_T = t^{-1}(t_a, t_b)$ globally hyperbolic time strip with timelike boundary
- S is a Friedrich systems with admissible boundary condition B_{adm}
- $\mathsf{S}^\dagger$ is the formal adjoint of S with admissible boundary condition $\mathsf{B}^\dagger_{adm}$
- (Σ is compact if S is a symmetric hyperbolic system)

Then $\exists\ \widetilde{C} = \widetilde{C}(M_T) > 0$ s. t. $\forall\ \Phi$ s.t. $\Phi|_{\Sigma_{t_a}} = 0$, $\Phi|_{\Sigma_{t_b}} = 0$ and $\Phi \in \mathsf{B}^\dagger_{adm}$

$$\|\Phi\|_{L^2(E_{|_{M_T}})} \leq \widetilde{C} \|\mathsf{S}^\dagger \Phi\|_{L^2(E_{|_{M_T}})}$$

Sketch of the proof:

$$2(\mathsf{S}^\dagger\Phi\,|\,\Phi)_{M_T} + (\Phi\,|\,\sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi)_{\partial M_T} = (\Phi\,|\,\mathsf{S}\Phi)_{M_T} - (\mathsf{S}^\dagger\Phi\,|\,\Phi)_{M_T} + 2(\mathsf{S}^\dagger\Phi\,|\,\Phi)_{M_T} \geq c(\Phi\,|\,\Phi)_{M_T}$$

since $\Phi \in \mathsf{B}^\dagger \Longrightarrow (\Phi\,|\,\sigma_{\mathsf{S}}(\mathtt{n})\Phi)_{\partial M_T} \leq 0 \Longrightarrow c(\Phi\,|\,\Phi)_{M_T} \leq 2(\Phi\,|\,\mathsf{S}^\dagger\Phi)_{M_T}$

WEAK and STRONG solutions in a time strip $M_T := t^{-1}(t_a, t_b)$

Definition: We call $\Psi \in \mathscr{H} := \overline{(\Gamma_c(E|_{M_T}), (.\,|\,.)_{M_T})}^{(.\,|\,.)_{M_T}}$

(W) **Weak Solution** if it holds $(\Phi\,|\,\mathfrak{f})_{M_T} = (\mathsf{S}^\dagger\Phi\,|\,\Psi)_{M_T}$

for any $\Phi \in \Gamma_c(E|_{M_T})$ such that $\Phi \in \mathsf{B}^\dagger_{adm}$ and $\Phi|_{\Sigma_{t_1}} = 0 = \Phi|_{\Sigma_{t_0}}$

(S) **Strong Solution** if $\exists\,\{\Psi_k\}_k$, $\Psi_k \in \Gamma(E|_{M_T})$ s.t. $\Psi_k \in \mathsf{B}_{adm}$ on ∂M and

$$\|\Psi_k - \Psi\|_{L^2(M_T)} \xrightarrow{k\to\infty} 0 \quad \text{and} \quad \|\mathsf{S}\Psi_k - \mathfrak{f}\|_{L^2(M_T)} \xrightarrow{k\to\infty} 0$$

Theorem: Any weak solution is a strong solution if

- $M_T = t^{-1}(t_a, t_b)$ globally hyperbolic time strip (Σ compact for S.H.S.)
- S is a Friedrich systems with admissible boundary condition B_{adm}

Comments on the Proof

- Admissible boundary conditions are local, so we can localise
- In Fermi coordinates, we can use the local theory [Phillips-Lax,Rauch,Massey-Rauch].

WEAK and STRONG solutions in a time strip $M_T := t^{-1}(t_a, t_b)$

Theorem: There exists a unique weak solution if

- $M_T = t^{-1}(t_a, t_b)$ globally hyperbolic time strip (Σ compact for S.H.S.)
- S is a Friedrich systems with admissible boundary condition B_{adm}

Sketch of the proof:

- Energy Estimates: $\|\Phi\|_{L^2(M_T)} \leq c\|\mathsf{S}^\dagger\Phi\|_{L^2(M_T)}$
- The kernel of the operator $\mathsf{S}^\dagger$ acting on $\operatorname{dom}\mathsf{S}^\dagger$ is trivial

$$\operatorname{dom}\mathsf{S}^\dagger := \{\Phi \in \Gamma_c(E_{M_T}) \mid \Phi|_{\Sigma_{t_1}} = 0, \Phi|_{\Sigma_{t_0}} = 0, \Phi \in \mathsf{B}_{adm}\}$$

- $\ell\colon \mathsf{S}^\dagger(\operatorname{dom}\mathsf{S}^\dagger) \to \mathbb{C}$ given by $\ell(\Theta) = (\Phi \mid \mathfrak{f})_{M_T}$ where Φ satisfies $\mathsf{S}^\dagger\Phi = \Theta$
- Energy Estimates $\Rightarrow$ ℓ is bounded:

$$\ell(\Theta) = (\Phi \mid \mathfrak{f})_{M_T} \leq \|\mathfrak{f}\|_{L^2(M_T)}\|\Phi\|_{L^2(M_T)} \qquad \text{(Cauchy-Schwarz inequality)}$$
$$\leq \lambda^{-1}\|\mathfrak{f}\|_{L^2(M_T)}\|\mathsf{S}^\dagger\Phi\|_{L^2(M_T)} = \lambda^{-1}\|\mathfrak{f}\|_{L^2(M_T)}\|\Theta\|_{L^2(M_T)},$$

-Hence $(\Phi \mid \mathfrak{f})_{M_T} = \ell(\Theta) \overset{Riesz}{\underset{Thm.}{=\!=}} (\Theta \mid \Psi)_{M_T} = (\mathsf{S}^\dagger\Phi \mid \Psi)_{M_T}$ for all $\Phi \in \operatorname{dom}\mathsf{S}^\dagger$

□

Differentiability for Friedrichs systems with $\sigma_{\mathsf{S}}(dt) > 0$

- Friedrichs system $\mathsf{S} = \sigma_{\mathsf{S}}(dt)\nabla_t - H$ with $\sigma_{\mathsf{S}}(dt) > 0$ and $\mathsf{B} = \ker G_{\mathsf{B}}$
- The compatibility condition of order $k \geq 0$ for $\mathfrak{h} \in \Gamma(E_{|\Sigma_{t_0}})$ and $\mathfrak{f} \in \Gamma(E)$ reads

$$\sum_{j=0}^{k} \frac{(k)!}{j!(k-j)!} \left(\nabla_t^j G_{\mathsf{B}}\right)\Big|_{\partial\Sigma_0} \mathfrak{h}_{k-j} = 0, \tag{1}$$

where the sequence $(\mathfrak{h}_k)_k$ of sections of $E_{|\partial\Sigma_0}$ is defined inductively by $\mathfrak{h}_0 := \mathfrak{h}$ and

$$\mathfrak{h}_k := \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-1-j)!} H_{j|\partial\Sigma_0} \mathfrak{h}_{k-1-j} + \nabla_t^{k-1}(\sigma_{\mathsf{S}}^{-1}(dt)\mathfrak{f})_{|\partial\Sigma_0} \qquad \text{for all } k \geq 1,$$

where $H_j := [\nabla_t, H_{j-1}]$ and $H_0 := \sigma_{\mathsf{S}}(dt)^{-1} H$

Theorem (N. Ginoux - S. M.) – TAKE HOME MESSAGE

- M is globally hyperbolic manifold with timelike boundary
- S is a Friedrich systems with $\sigma_{\mathsf{S}}(dt) > 0$ with admissible boundary condition B_{adm}
- Cauchy data satisfies the compatibility condition (1)

The Cauchy problem is well-posed

Outlook

WHAT WE KNOW AND WHAT COMES NEXT?

well-posedness for symmetric hyperbolic system with B_{adm}

✓ Classical Dirac operator with Chiral and MIT boundary conditions

✓ Wave equation with Neumann and transparent boundary condition

What comes next?

- Propagation of singularities for symmetric hyperbolic systems with B_{adm}
- Well-posedness and support properties for Klein-Gordon operator with B_{adm}^{Robin}
- Symmetric hyperbolic systems with nonlocal boundary conditions

  e.g. Dirac operator with APS boundary conditions
- Friedrichs systems with constraints

  e.g. Maxwell equations or Euler equation for incompressible fluids

THANKS for your attention!