On the Cauchy problem for symmetric hyperbolic systems on globally hyperbolic manifolds with timelike boundary

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Outline of the Talk

- **Preliminaries**
  - **Globally hyperbolic manifolds with timelike boundary**
  - **Symmetric hyperbolic systems**
- **Boundary conditions**
  - **Admissible boundary conditions**
  - **Self-adjoint elliptic boundary conditions**
- **Energy estimates**
- **Existence and uniqueness of smooth solutions**
- **Outlook**

Based on

S. M. and N. Ginoux *"On the Cauchy problem for Friedrichs systems on globally hyperbolic manifolds with timelike boundary"* arXiv:2007.02544

N. Drago, N. Große and S.M.*"Well-posedness of the initial-boundary value problem for the classical Dirac operator with selfadjoint elliptic boundary conditions"*

C. Bär *"Geometric wave equations"* Geometry in Potsdam (2017)

Globally hyperbolic manifolds with timelike boundary

M is connected, time-oriented, oriented smooth manifold with timelike boundary ∂M

Few important definitions:

- *Temporal function:* $t \in C^\infty(M, \mathbb{R})$ strictly increasing on future directed causal curve and ∇t is timelike everywhere and past-pointing
- *Cauchy hypersurface* Σ: if each inextendible timelike curve $\gamma \cap \Sigma = \{\mathrm{pt}\}$
- *Globally hyperbolic*: M *strongly causal* and $\forall p, q \in M$,$J^+(p) \cap J^-(q)$ compact

Bernal and Sánchez (2005) – Aké, Flores and Sánchez (2019):

M is globally hyperbolic (with timelike boundary)

$$\Updownarrow$$

Exists a Cauchy temporal function ($t^{-1}(s) := \Sigma_s$ is a Cauchy) and $\nabla t \in T\partial M$

$$\Downarrow$$

M isometric to $\mathbb{R} \times \Sigma$ with metric $-\beta^2 dt^2 + h_t$, where $\beta \in C^\infty(M, (0, \infty))$

Example: Minkoski spacetime $(\mathbb{R}^4, \eta)$, Schwarzchild spacetime $(\mathbb{R}^2 \times \mathbb{S}^2, g_S)$

NOT Example: anti-de Sitter space $(\mathbb{S}^1 \times \mathbb{R}^3, g_{adS})$, Gödel universe $(\mathbb{R}^4, g_G)$

Symmetric hyperbolic systems

- $E \to M$ be a $\mathbb{K}$-vector bundle with finite rank N and sesquilinear fiber metric $\prec \cdot | \cdot \succ$

Definition: a 1^{st} order S is called **symmetric hyperbolic system** if

(S) $\sigma_S(\xi)\colon E_p \to E_p$ is Hermitian with respect to $\prec \cdot | \cdot \succ$, $\forall \xi \in T_p^* M$ and $\forall p \in M$.

(H) $\prec \sigma_S(\tau) \cdot \, | \cdot \succ$ is positive definite on E_p, for any future-directed timelike $\tau \in T_p^* M$

If $\dim \ker \sigma_S(\mathtt{n}^\flat)$ is constant then S is **of constant characteristic** (where $\mathtt{n} \perp \partial M$).

Example: $E = \mathbb{C}^N \times \mathbb{R}^n \to (\mathbb{R}^n, \eta)$ with $\prec \, | \, \succ = \langle \, | \, \rangle_{\mathbb{C}^N}$

$$\mathsf{S} := A_0(p)\partial_t + \sum_{j=i}^{n} A_j(p)\partial_{x_j} + B(p)$$

(S) $A_0 = A_0^\dagger \,,\ A_j = A_j^\dagger$ $\qquad$ (H) $\sigma_S(dt + \sum_j \alpha_j dx_j) = A_0 + \sum_{j=1} \alpha_j A_j > 0\,.$

Lemma [Ginoux-M.]: If $\prec \cdot | \cdot \succ$ is indefinite and S is a symmetric hyperbolic system:

(I) $\langle \cdot | \cdot \rangle := \prec \sigma_S(dt) \cdot \, | \cdot \succ$ is positive Hermitian metric;

(II) $\mathfrak{S} = -\sigma_S(dt)^{-1}\mathsf{S}$. is symmetric hyperbolic system

(III) Cauchy problem for $\mathfrak{S}$ is equivalent to the Cauchy problem for S.

Example I: The classical Dirac operator

- $M = (M, g)$ is a globally hyperbolic spin manifold with timelike boundary;
- $\mathbb{S}M$ is a *spinor bundle*: $\mathbb{C}$-vector bundle with indefinite sesquilinear metric

$$\prec \cdot \,|\, \cdot \succ \colon \mathbb{S}_p M \times \mathbb{S}_p M \to \mathbb{C}$$

  and a Clifford multiplication, i.e. fiber-preserving map $\gamma \colon TM \to \mathrm{End}(\mathbb{S}M)$

Dirac operator: $\mathrm{D} := \gamma \circ \nabla^{\mathbb{S}} \colon \Gamma(\mathbb{S}M) \to \Gamma(\mathbb{S}M)$ which in local coordinates reads

$$\mathrm{D} = \sum_{\mu=0}^{n} \varepsilon_\mu \gamma(e_\mu) \nabla^{\mathbb{S}}_{e_\mu}$$

- $(e_\mu)_{\mu=0,\dots,n}$ is a local orthonormal Lorentzian frame of TM and $\varepsilon_\mu := g(e_\mu, e_\mu)$
- $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u, v)$ for every $u, v \in T_pM$ and $p \in M$.

Remarks:

(i) Topological obstruction to existence of a spinor bundle;

(ii) Existence of spinor bundles on parallelizable manifolds;

(iii) D is nowhere characteristic.

Example II: The geometric wave operator

-M is a globally hyperbolic with timelike boundary and $g = -\beta^2 dt^2 + h_t$;

- V be an Hermitian vector bundle of finite rank;

- P is a normally hyperbolic operator, i.e. $P = \nabla^*\nabla + c$ and principal symbol σ_P defined by

$$\sigma_P(\xi) = -g(\xi^\sharp, \xi^\sharp) \cdot \mathrm{Id}_V \,, \qquad \text{for every } \xi \in T^*M.$$

A norm. hyp. op. P can be reduced to $\mathsf{S} : \Gamma(E) \to \Gamma(E)$ defined by

$$\mathsf{S} := (A_0 \nabla_{\partial_t} + A_\Sigma \nabla^\Sigma + C)$$

$$\Psi := \begin{pmatrix} \nabla_{\partial_t} u \\ \nabla^\Sigma u \\ u \end{pmatrix} \quad A_0 := \begin{pmatrix} \frac{1}{\beta^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_\Sigma = \begin{pmatrix} 0 & -\mathrm{tr}_{h_t} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} \text{suitable} \end{pmatrix}$$

Remarks:

(i) The Cauchy problem for P can be made equivalent to the Cauchy problem for S;

(ii) S is of constant characteristic:

$$\sigma_S(\mathtt{n}^\flat) = \begin{pmatrix} 0 & -\mathtt{n}^\flat \lrcorner & 0 \\ -\mathtt{n}^\flat \otimes & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Admissible boundary conditions

Definition [Friedrichs]: A boundary space B_{adm} for S is called **admissible** if

- The quadratic form $\Psi \mapsto \prec \sigma_S(\mathtt{n}^\flat)\Psi \,|\, \Psi \succ$ is positive semi-definite on B_{adm};
- $\operatorname{rank} \mathsf{B}_{adm} = \#$ pointwise non-negative eigenvalues of $\sigma_S(\mathtt{n}^\flat)$ counting multiplicity.

The adjoint boundary space is defined by $\mathsf{B}^\dagger_{adm} := \big(\sigma_\mathsf{S}(\mathtt{n}^\flat)(\mathsf{B}_{adm})\big)^\perp$, i.e.

$$\{\Phi \in \Gamma(E|_{\partial M}) \,|\, \text{for any } \Psi \in \mathsf{B}_{adm} \text{ it holds } \prec \sigma_\mathsf{S}(\mathtt{n}^\flat)\Psi \,|\, \Phi \succ = 0\}\,.$$

Examples for classical Dirac operators:

Lorentzian MIT boundary space is the range of $\pi_{\mathrm{Lor}} := \frac{1}{2}\left(\mathrm{Id} \pm \imath\gamma(\mathtt{n})\right)$

$$\prec \sigma_\mathsf{D}(\mathtt{n}^\flat)\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ = \prec \gamma(\mathtt{n})\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ = \imath \prec \pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ = 0$$

Riemannian MIT boundary space is the range of $\pi_{\mathrm{Riem}} := \frac{1}{2}\left(\mathrm{Id} - \frac{1}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\right)$

$$\begin{aligned}\prec \sigma_\mathsf{D}(\mathtt{n}^\flat)\pi_{\mathrm{Riem}}\psi \,|\, \pi_{\mathrm{Riem}}\psi \succ &= \prec \gamma(\mathtt{n})\pi_{\mathrm{Riem}}\psi \,|\, \pi_{\mathrm{Riem}}\psi \succ \\ &= \frac{1}{\beta} \prec \gamma(\partial_t)\pi_{\mathrm{Riem}}\psi \,|\, \pi_{\mathrm{Riem}}\psi \succ \geq 0\end{aligned}$$

Admissible boundary conditions

Definition [Friedrichs]: A boundary space B_{adm} for S is called **admissible** if

- The quadratic form $\Psi \mapsto \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \,|\, \Psi \succ$ is positive semi-definite on B_{adm};
- $\operatorname{rank} \mathsf{B}_{adm} = \#$ pointwise non-negative eigenvalues of $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ counting multiplicity.

The adjoint boundary space is defined by $\mathsf{B}^\dagger_{adm} := \big(\sigma_{\mathsf{S}}(\mathtt{n}^\flat)(\mathsf{B}_{adm})\big)^\perp$, i.e.

$$\{\Phi \in \Gamma(E|_{\partial M}) \,|\, \text{for any } \Psi \in \mathsf{B}_{adm} \text{ it holds } \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \,|\, \Phi \succ = 0\}\,.$$

Examples for geometric wave operator:

Neumann like-boundary condition: $\nabla^\Sigma_{\mathtt{n}} u = 0 \implies \mathsf{B}^{N.}_{adm} = \ker \begin{pmatrix} 0 & \mathtt{n}\lrcorner & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Transparent boundary condition: $\nabla^\Sigma_{\mathtt{n}} u = -b\nabla_{\partial_t} u \implies \mathsf{B}^{T.}_{adm} = \ker \begin{pmatrix} b & \mathtt{n}\lrcorner & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

NOT example for geometric wave operator:

Robin boundary condition: $\nabla^\Sigma_{\mathtt{n}} u = bu \quad (b \neq 0) \Rightarrow \mathsf{B}^{R.}_{adm} = \ker \begin{pmatrix} 0 & \mathtt{n}\lrcorner & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(the quadratic form is not positive semi-definite)

Self-adjoint elliptic boundary conditions

- M is a globally hyperbolic spin with timelike boundary and $\partial\Sigma$ is compact;
- Dirac operator on spinor bundle $\mathbb{S}M$ reads as $\mathsf{D} = -\gamma(dt)\nabla_{\partial_t} + \mathsf{D}_\Sigma$
- Dirac Hamiltonian $\mathsf{H} := \imath\gamma(dt)^{-1}\mathsf{D}_\Sigma \qquad (\mathsf{D}\Psi = 0 \iff \imath\nabla_{\partial_t}\Psi = \mathsf{H}\Psi)$
- Adapted operator $A : \Gamma(\mathbb{S}M|_{\partial M}) \to \Gamma(\mathbb{S}M|_{\partial M})$ formally self-adjoint

$$\sigma_A(\xi) := \sigma_{\mathsf{H}(t)}(\nu_t^\flat)^{-1} \circ \sigma_{\mathsf{H}(t)}(\xi) \qquad \sigma_{\mathsf{H}(t)}(\nu_t^\flat) \circ A = -A \circ \sigma_{\mathsf{H}(t)}(\nu_t^\flat)\,,$$

- Sobolev-like spaces $\qquad \check{\mathcal{H}}(A(t)) := \mathcal{H}^{\frac{1}{2}}_{(-\infty,a)}(A(t)) \oplus \mathcal{H}^{-\frac{1}{2}}_{[a,\infty)}(A(t))$

$$\mathcal{H}^s_I(A(t)) := \left\{ \sum_j \alpha_j\varphi_j(t) \in L^2(\mathbb{S}M|_{\partial\Sigma_t}) \,\middle|\, \sum_{j|\lambda_j\in I} |\alpha_j|^2(1+\lambda_j^2(t))^s < +\infty \right\}, \quad s \in \mathbb{R}$$

Definition [Bär-Ballmann]: A boundary space B_{ell} for D is called **self-adjoint elliptic** if

(B) $\mathsf{B}_{ell}(t) \subset \check{\mathcal{H}}(A(t))$ is closed;

(E) $\Psi \in \mathcal{H}^k_{loc}(\mathbb{S}M|_{\Sigma_t}) \iff \mathsf{H}_{ell}(t)\Psi \in \mathcal{H}^{k+1}_{loc}(\mathbb{S}M|_{\Sigma_t})\,,$

(SA) $\mathsf{B}_{ell}(t) = \mathsf{B}^\dagger_{ell}(t) := \{\Phi \in \check{\mathcal{H}}(A(t)) \,|\, \forall\Psi \in \mathsf{B}_{ell} \text{ it holds } \prec \sigma_{\mathsf{H}(t)}(\mathtt{n}^\flat)\Psi \,|\, \Phi \succ = 0\}$

Examples: $\mathsf{B}_{APS} := \mathcal{H}^{\frac{1}{2}}_{(-\infty,0)}(A(t)) \quad$ (self-adjoint $\iff$ $\ker A(t) = \{0\}$).

TAKE HOME MESSAGE: The Cauchy problem is well-posed

Theorem [Ginoux-M.] The Cauchy problem is well-posed for

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary
- S is a symmetric hyperbolic system with admissible boundary conditions B_{adm}
- $(\mathfrak{h}, \mathfrak{f})$ are Cauchy data satisfying compatibility condition up to any order

Remarks: in [Ginoux-M.] it also been proved

(i) Existence and uniqueness of strong solutions for Friedrichs systems (e.g. Heat equation and Klein-Gordon equation)

(ii) Robin boundary conditions are admissible for Heat and KG-equations

Theorem [Drago-Große-M.] The Cauchy problem is well-posed for:

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary with $\partial\Sigma$ compact
- D is the classical Dirac operator with APS boundary conditions B_{APS}
- $(\mathfrak{h}, \mathfrak{f})$ are *suitable* Cauchy data satisfying compatibility condition up to any order

Remarks: in [Drago, Große, M.] we expect

(i) well-posedness for "*continuous*" self-adjoint elliptic boundary conditions, i.e.

$$t \mapsto \| J_t^{(\varepsilon)} \hat{\mathrm{H}}_{t,\hat{B}_t} J_t^{(\varepsilon)} - i J_t^{(\varepsilon)} (\partial_t \tau_{-t}^* \Lambda) J_t^{(\varepsilon)} \|_{\mathcal{B}(\mathcal{H}^k_{\hat{B}_t}(S\mathcal{M}|_\Sigma))} \text{ is continuous}$$

Energy estimates

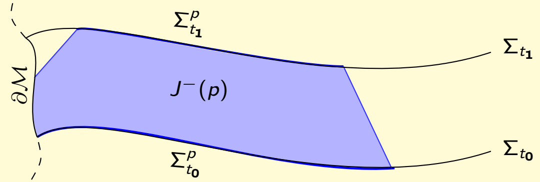
Theorem (Energy estimates)

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary
- S is symmetric hyperbolic with B_{adm}

Then for each $t_0 \in t(M)$ $\exists$ $C > 0$ s. t. $\forall$ $t_1 \geq t_0$ it holds $\forall\, \Psi \in \mathsf{B}_{adm}$

$$\int_{\Sigma^p_{t_1}} |\Psi|^2 d\mu_{t_1} \leq Ce^{C(t_1 - t_0)} \int_{t_0}^{t_1} \int_{\Sigma^p_s} |\mathsf{S}\Psi|^2 d\mu_s ds + e^{C(t_1 - t_0)} \int_{\Sigma_{t^p_0}} |\Psi|^2 d\mu_{t_0}$$

where $t\colon M \to \mathbb{R}$ be a Cauchy temporal function and $\Sigma^p_s := J^-(p) \cap \Sigma_s$



Energy estimates

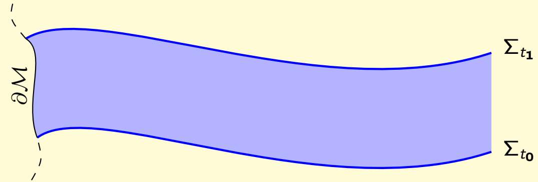
Theorem (Energy estimates)

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary and $\partial\Sigma$ compact
- D is Dirac operator with B_{ell}

Then for each $t_0 \in t(M)$ $\exists$ $C > 0$ s. t. $\forall$ $t_1 \geq t_0$ it holds $\forall \Psi \in \mathsf{B}_{ell}$

$$\int_{\Sigma_{t_1}} |\Psi|^2 d\mu_{t_1} \leq C e^{C(t_1 - t_0)} \int_{t_0}^{t_1} \int_{\Sigma_s} |\mathsf{D}\Psi|^2 d\mu_s ds + e^{C(t_1 - t_0)} \int_{\Sigma_{t_0}} |\Psi|^2 d\mu_{t_0}$$

where $t\colon M \to \mathbb{R}$ be a Cauchy temporal function



Energy estimates

Theorem (Energy estimates)

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary (and $\partial\Sigma$ compact for B_{ell})
- S is symmetric hyperbolic with B_{adm} while D is Dirac operator with B_{ell}

Then for each $t_0 \in t(M)$ there exists constants $C > 0$ such that for all $t_1 \geq t_0$ it holds

$$\int_{\Sigma_{t_1}} |\Psi|^2 d\mu_{t_1} \leq Ce^{C(t_1 - t_0)} \int_{t_0}^{t_1} \int_{\Sigma_s} |\mathsf{D}\Psi|^2 d\mu_s ds + e^{C(t_1 - t_0)} \int_{\Sigma_{t_0}} |\Psi|^2 d\mu_{t_0} \quad \forall \Psi \in \mathsf{B}_{ell}$$

$$\int_{\Sigma^p_{t_1}} |\Psi|^2 d\mu_{t_1} \leq Ce^{C(t_1 - t_0)} \int_{t_0}^{t_1} \int_{\Sigma^p_s} |\mathsf{S}\Psi|^2 d\mu_s ds + e^{C(t_1 - t_0)} \int_{\Sigma_{t^p_0}} |\Psi|^2 d\mu_{t_0} \quad \forall \Psi \in \mathsf{B}_{adm}$$

where $t\colon M \to \mathbb{R}$ be a Cauchy temporal function and $\Sigma^p_s := J^-(p) \cap \Sigma_s$

Corollary (uniqueness)

If there exists a solution to the Cauchy problem with B_{adm} or B_{ell}, then it is **unique**

Proof: If $\mathsf{S}(\Psi - \Phi) = 0$, $\Psi, \Phi \in \mathsf{B}_{adm/ell}$ *and* $\Psi|_{\Sigma_0} = \Phi|_{\Sigma_0} \xrightarrow{\text{Energy estimates}} \Psi - \Phi = 0$

Energy estimates

Theorem (Energy estimates)

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary (and $\partial\Sigma$ compact for B_{ell})
- S is symmetric hyperbolic with B_{adm} while D is Dirac operator with B_{ell}

Then for each $t_0 \in t(M)$ there exists constants $C > 0$ such that for all $t_1 \geq t_0$ it holds

$$\int_{\Sigma_{t_1}} |\Psi|^2 d\mu_{t_1} \leq Ce^{C(t_1 - t_0)} \int_{t_0}^{t_1} \int_{\Sigma_s} |D\Psi|^2 d\mu_s ds + e^{C(t_1 - t_0)} \int_{\Sigma_{t_0}} |\Psi|^2 d\mu_{t_0} \quad \forall \Psi \in B_{ell}$$

$$\int_{\Sigma^p_{t_1}} |\Psi|^2 d\mu_{t_1} \leq Ce^{C(t_1 - t_0)} \int_{t_0}^{t_1} \int_{\Sigma^p_s} |S\Psi|^2 d\mu_s ds + e^{C(t_1 - t_0)} \int_{\Sigma_{t^p_0}} |\Psi|^2 d\mu_{t_0} \quad \forall \Psi \in B_{adm}$$

where $t \colon M \to \mathbb{R}$ be a Cauchy temporal function and $\Sigma^p_s := J^-(p) \cap \Sigma_s$

Corollary (uniqueness)

If there exists a solution to the Cauchy problem with B_{adm} or B_{ell}, then it is **unique**

Corollary (finite speed of propagation with B_{adm} or with B_{adm})

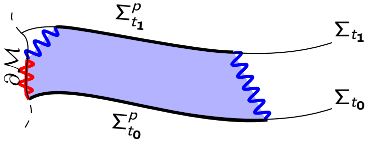
$$\text{supp } \Psi \subset \mathcal{V} := J(\text{supp } f) \cup J(\text{supp } h) \cup J(\partial\Sigma_0)$$

Proof: If $p \notin \mathcal{V}$ *then* $\mathfrak{f}|_{M\setminus\mathcal{V}} = 0$ *and* $\mathfrak{h}|_{M\setminus\mathcal{V}} = 0 \xrightarrow{\text{Energy estimates}} \Psi|_{M\setminus\mathcal{V}} = 0$

Energy estimates (admissible boundary conditions)

Sketch of the proof:

- n-differential form:



$$\omega := \sum_{j=0}^n \Re e \left(\prec \sigma_{\mathsf{S}}(b_j^\flat)\Psi \mid \Psi \succ \right) b_j \lrcorner \mathrm{vol}_M$$

- Stokes' theorem for manifold with Lipschitz boundary yields

$$\int_K d\omega = \int_{\partial K} \omega = \int_{\Sigma^p_{t_1}} \omega - \int_{\Sigma^p_{t_0}} \omega + \int_{\text{red}} \omega + \int_{\text{blue}} \omega$$

- Hyperbolicity of S $\Longrightarrow \int_{\text{blue}} \omega \geq 0$ while $\Psi \in \mathsf{B}_{adm} \Longrightarrow \int_{\text{red}} \omega \geq 0$

$$\int_{\Sigma^p_{t_1}} |\Psi|^2 d\mu_1 - \int_{\Sigma^p_{t_0}} |\Psi|^2 d\mu_0 \leq \int_K d\omega \leq C \int_{t_0}^{t_1} \int_{\Sigma^p_s} (|\Psi|^2 + |\mathsf{S}\Psi|^2) d\mu_s ds$$

- By setting $h(s) := \int_{\Sigma^p_s} |\Psi|^2 d\mu_s$, $\quad \alpha(t_1) := C \int_{t_0}^{t_1} \int_{\Sigma^p_s} (|\Psi|^2 + |\mathsf{S}\Psi|^2) d\mu_s ds + \int_{\Sigma^p_0} |\Psi|^2 d\mu_0$

and using Grönwall, we obtain: $h(t_1) \leq \alpha(t_1) + C \int_{t_0}^{t_1} h(s) ds \leq \alpha(t_1) e^{C(t_1 - t_0)}$

□

WEAK and STRONG solutions in a time strip $M_T := t^{-1}(t_1, t_0)$

Definition: We call $\Psi \in \mathscr{H} := \overline{\left(\Gamma_c(E|_{M_T}), (.\,|\,.)_{M_T}\right)}^{(.\,|\,.)_{M_T}}$

(W) **Weak Solution** if it holds $(\Phi\,|\,\mathfrak{f})_{M_T} = (\mathsf{S}^\dagger\Phi\,|\,\Psi)_{M_T}$

for any $\Phi \in \Gamma_c(E|_{M_T})$ such that $\Phi \in \mathsf{B}^\dagger_{adm/ell}$ and $\Phi|_{\Sigma_{t_1}} = 0 = \Phi|_{\Sigma_{t_0}}$

(S) **Strong Solution** if $\exists\,\{\Psi_k\}_k$, $\Psi_k \in \Gamma(E|_{M_T})$ s.t. $\Psi_k \in \mathsf{B}_{adm/ell}$ on ∂M and

$$\|\Psi_k - \Psi\|_{L^2(M_T)} \xrightarrow{k\to\infty} 0 \quad \text{and} \quad \|\mathsf{S}\Psi_k - \mathfrak{f}\|_{L^2(M_T)} \xrightarrow{k\to\infty} 0$$

Theorem (weak=strong with admissible boundary conditions B_{adm})

- $M = \mathbb{R}\times\Sigma$ globally hyperbolic with timelike boundary
- S is symmetric hyperbolic with admissible boundary conditions B_{adm}

Any weak solution is a strong solution.

Moreover, if $(\mathfrak{f}, \mathfrak{h})$ satisfy compatibility conditions the solution is **smooth**.

Comments on the Proof

- Admissible boundary conditions are local, so we can localise
- In Fermi coordinates, we can use the local theory [Phillips-Lax,Rauch,Massey-Rauch].

Existence of WEAK solutions with B_{adm} or B_{ell}

Theorem (existence weak solutions)

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary (and $\partial\Sigma$ compact for B_{ell})
- S is symmetric hyperbolic with B_{adm} or is Dirac operator with B_{ell}

There exists a weak solution to the Cauchy problem.

Sketch of the proof:

- Energy Estimates: $\|\Phi\|_{L^2(M_T)} \leq c \|\mathsf{S}^\dagger \Phi\|_{L^2(M_T)}$
- The kernel of the operator $\mathsf{S}^\dagger$ acting on $\operatorname{dom} \mathsf{S}^\dagger$ is trivial

$$\operatorname{dom} \mathsf{S}^\dagger := \{\Phi \in \Gamma_c(E_{M_T}) \mid \Phi|_{\Sigma_{t_1}} = 0, \Phi|_{\Sigma_{t_0}} = 0, \Phi \in \mathsf{B}_{adm/ell}\}$$

- $\ell \colon \mathsf{S}^\dagger(\operatorname{dom} \mathsf{S}^\dagger) \to \mathbb{C}$ given by $\ell(\Theta) = (\Phi \mid \mathfrak{f})_{M_T}$ where Φ satisfies $\mathsf{S}^\dagger \Phi = \Theta$
- Energy Estimates $\Rightarrow \ell$ is bounded:

$$\begin{aligned} \ell(\Theta) = (\Phi \mid \mathfrak{f})_{M_T} &\leq \|\mathfrak{f}\|_{L^2(M_T)} \|\Phi\|_{L^2(M_T)} \qquad \text{(Cauchy-Schwarz inequality)} \\ &\leq \lambda^{-1} \|\mathfrak{f}\|_{L^2(M_T)} \|\mathsf{S}^\dagger \Phi\|_{L^2(M_T)} = \lambda^{-1} \|\mathfrak{f}\|_{L^2(M_T)} \|\Theta\|_{L^2(M_T)}, \end{aligned}$$

-Hence $(\Phi \mid \mathfrak{f})_{M_T} = \ell(\Theta) \overset{Riesz}{\underset{Thm.}{=\!=}} (\Theta \mid \Psi)_{M_T} = (\mathsf{S}^\dagger \Phi \mid \Psi)_{M_T}$ for all $\Phi \in \operatorname{dom} \mathsf{S}^\dagger$

□

Existence of SMOOTH solutions with B_{APS}

Theorem (existence smooth solutions with APS boundary conditions B_{APS})

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary and $\partial\Sigma$ compact
- D is a Dirac operator with APS boundary conditions B_{APS}

There exists a **unique smooth solutions** to the Cauchy problem.

Sketch of the proof

- By finite speed of propagation in the bulk, Σ can be chosen compact
- Identification (part I)

$$\Gamma(SM) \Longleftrightarrow C^\infty(\mathbb{R}, \Gamma(SM|_\Sigma)) \quad \text{s.t.} \quad L^2(\Gamma(SM|_{\Sigma_t})) \Longleftrightarrow L^2(\Gamma(SM|_\Sigma))$$

- Identification (part II)

$$\begin{cases} \mathrm{D}\psi = f\,, \\ \psi|_\Sigma = \psi_0\,, \\ \psi|_{\partial\Sigma_t} \in B_t \end{cases} \Longleftrightarrow \begin{cases} i\partial_t \Psi_t = \widehat{\mathrm{H}}_t \Psi_t - i(\partial_t \tau^*_{-t}\Lambda)\Psi_t + \hat{f}_t \\ \Psi_0 = \psi_0\,, \\ \Psi_t \in \hat{B}_t\,, \end{cases}$$

- Sobolev-like spaces $\mathcal{H}^\infty_{APS_t}(SM|_\Sigma) := \bigcap_k \mathrm{dom}(\hat{\mathrm{H}}^2_{APS_t} + 1)^{\frac{k}{2}} \subset \Gamma(SM|_\Sigma)$

– Mollifier $J_\varepsilon(t) := \exp[-\varepsilon\langle \hat{\mathrm{H}}_{APS_t}(t)^2 + 1\rangle] \colon \mathcal{H}^\infty_{APS_t}(SM|_\Sigma) \to \mathcal{H}^\infty_{APS_t}(SM|_\Sigma)$

Existence of SMOOTH solutions with B_{APS}

Theorem (existence smooth solutions with APS boundary conditions B_{APS})

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary and $\partial\Sigma$ compact
- D is a Dirac operator with APS boundary conditions B_{APS}

There exists a **unique smooth solutions** to the Cauchy problem.

Sketch of the proof

- Existence of unique solution of regularized equation

$$\imath\partial_t \Psi_t^{(\varepsilon)} = J_t^{(\varepsilon)} \hat{H}_{APS_t} J_t^{(\varepsilon)} \Psi_t^{(\varepsilon)} - iJ_t^{(\varepsilon)}(\partial_t \tau_{-t}^* \Lambda) J_t^{(\varepsilon)} \Psi_t^{(\varepsilon)} + f\,,$$

(here we used $t \mapsto \| J_t^{(\varepsilon)} \hat{H}_{APS_t} J_t^{(\varepsilon)} - iJ_t^{(\varepsilon)}(\partial_t \tau_{-t}^* \Lambda) J_t^{(\varepsilon)} \|_{\mathcal{B}(\mathcal{H}^k_{\hat{B}_t}(S\mathcal{M}|_\Sigma))}$ is continuous)

- Estimates + Ascoli-Arzela + diagonal subsequence argument

$$\{\Psi^{(\varepsilon)}\}_{\varepsilon>0} \supset \{\Psi^{(\varepsilon_j)}\}_{j\in\mathbb{N}} \xrightarrow{\varepsilon_j \searrow 0} \Psi \in C^0(\mathbb{R}, \mathcal{H}^\infty_{APS_t}(SM|_\Sigma))$$

and the regularized equation converges to

$$\imath\partial_t \Psi_t = \hat{H}_{APS_t} \Psi_t - i(\partial_t \tau_{-t}^* \Lambda)\Psi_t + f \quad \Longrightarrow \quad \Psi \in C^1(\mathbb{R}, \Gamma(SM|_\Sigma))$$

- By showing that also $\partial_t \Psi \in C^0(\mathbb{R}, \Gamma(SM|_\Sigma))$ and iterating, we can conclude. □

Outlook

WHAT WE HAVE SEEN AND WHAT COMES NEXT?

well-posedness of the Cauchy problem for

✓ Dirac operator with APS boundary conditions (Nicolò Drago & Nadine Große)

e.g. classical Dirac operator with APS boundary condition

✓ Symmetric hyperbolic systems with admissible b.c. (Nicolas Ginoux)

e.g. Wave equation with Neumann and transparent boundary condition

e.g. Classical Dirac operator with Chiral and MIT boundary conditions

What comes next?

? Index Theory for Dirac operator with APS boundary condition

? De Rham-d'Alembert operator with Robin boundary condition

? Symmetric Hyperbolic Systems with elliptic boundary condition

THANKS for your attention!