HADAMARD STATES FOR MAXWELL FIELDS VIA COMPLETELY GAUGE FIXING

Simone Murro

Department of Mathematics, University of Genoa

Thematic programme on

"*Spectral Theory and Mathematical Relativity*"

July 6, 2023

MOTIVATION

- Consider a *globally hyperbolic* 4-*manifold* $\mathrm{M} = (\mathbb{R} \times \Sigma, g = -\beta^2 dt^2 + h_t)$
- A *gauge theory* is a quadruple $(\mathrm{V}_0, \mathrm{V}_1, \mathrm{P}, \mathrm{K})$ consisting of:

(I) two Hermitian bundles $\mathrm{V}_0, \mathrm{V}_1$ over M;

(II) a formally self-adjoint differential operator $\mathrm{P} \colon \Gamma(\mathrm{V}_1) \to \Gamma(\mathrm{V}_1)$;

(III) a linear differential operator $\mathrm{K} \colon \Gamma(\mathrm{V}_0) \to \Gamma(\mathrm{V}_1)$ with $\mathrm{K} \neq 0$ such that

- (i) $\mathrm{P} \circ \mathrm{K} = 0$ *(gauge transformation)*
- (ii) $\mathrm{D}_1 := \mathrm{P} + \mathrm{K}\mathrm{K}^* \colon \Gamma(\mathrm{V}_1) \to \Gamma(\mathrm{V}_1)$ is Green hyperbolic;
- (iii) $\mathrm{D}_0 := \mathrm{K}^*\mathrm{K} \colon \Gamma(\mathrm{V}_0) \to \Gamma(\mathrm{V}_0)$ is Green hyperbolic,

Equivalently, we can work at the level of *initial data*

(I') V_{ρ_i} are the bundle of initial data for D_i

(III') $\mathrm{K}_\Sigma := \rho_1 K U_0 \quad \mathrm{K}_\Sigma^\dagger := \rho_0 K^* U_1 \quad \mathrm{G}_i = (\rho_i \mathrm{G}_i)^* \mathrm{G}_{i,\Sigma}(\rho_i \mathrm{G}_i)$

where ρ_i and U_i are the Cauchy data and the Cauchy evolution operator and $(\rho_i \mathrm{G}_i)^* : \Gamma_c(\mathrm{V}_{\rho_i}) \to \Gamma_{sc}(\mathrm{V}_i)$ is the adjoint of $(\rho_i \mathrm{G}_i)$

HOW CAN WE QUANTIZE IT?

Step 1: Construct the classical phase space

$$
\begin{array}{ccc}
\mathrm{i}(\cdot, \mathsf{G}_1 \cdot)_{\mathsf{V}_1} =: q_1 \,,\ \mathcal{V}_{\mathrm{P}} := \dfrac{\ker(\mathsf{K}^*|_{\Gamma_c})}{\operatorname{ran}(\mathsf{P}|_{\Gamma_c})} & \xrightarrow{\ [\mathsf{G}_1]\ } & \dfrac{\ker(\mathsf{P}|_{\Gamma_{\mathrm{sc}}})}{\operatorname{ran}(\mathsf{K}|_{\Gamma_{\mathrm{sc}}})} \\
\text{unitary} \Big\downarrow [\rho_1 \mathsf{G}_1] & \searrow^{[\mathsf{G}_1]} & \Big\uparrow [j] \\
\mathrm{i}(\cdot, \mathsf{G}_{1_\Sigma} \cdot)_{\mathsf{V}_{\rho_1}} =: q_{1_\Sigma} \,,\ \mathcal{V}_{\Sigma} := \dfrac{\ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_c})}{\operatorname{ran}(\mathsf{K}_\Sigma|_{\Gamma_c})} & \xrightarrow{\ [\mathcal{U}_1]\ } & \dfrac{\ker(\mathsf{D}_1|_{\Gamma_{\mathrm{sc}}}) \cap \ker(\mathsf{K}^*|_{\Gamma_{\mathrm{sc}}})}{\mathsf{K}(\ker(\mathsf{D}_0|_{\Gamma_{\mathrm{sc}}}))}
\end{array}
$$

and assign $\forall v \in \mathcal{V}_{\mathrm{P}}$ an element of the abstract unital $*$-algebra $\mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \mathrm{q}_1)$

generators: $\Phi(v) \qquad \Phi^*(v) \qquad \mathbb{1}$

CCR relations:
$$[\Phi(v), \Phi(w)] = [\Phi^*(v), \Phi^*(w)] = 0$$
$$[\Phi(v), \Phi^*(w)] = \mathrm{q}_1(v, w)\mathbb{1}$$

Step 2: Construct an **Hadamard states** $\omega : \mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \mathrm{q}_1) \to \mathbb{C}$ defined by

covariances: $\Lambda^+(v, w) := \omega(\Phi(v)\Phi^*(w)) \qquad \Lambda^-(v, w) := \omega(\Phi^*(w)\Phi(v))$

Hadamard conditions: $\mathrm{WF}'(\Lambda^\pm) \subset \mathcal{N}^\pm \times \mathcal{N}^\pm \qquad$ where: $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$

PROPOSITION [Gérard-Wrochna]: let $c^{\pm} : \Gamma_c(V_{\rho_1}) \to \Gamma(V_{\rho_1})$ be

(i) $c^{\pm}(\mathrm{ran}(K_\Sigma|_{\Gamma_c})) \subset \mathrm{ran}(K_\Sigma)$ and $(c^{\pm})^\dagger = c^{\pm}$ (w.r.t. $q_{1,\Sigma}$);

(ii) $(c^+ + c^-)\mathfrak{f} = \mathfrak{f}$ mod $\mathrm{ran}K_\Sigma$ $\forall \mathfrak{f} \in \ker(K_\Sigma^\dagger|_{\Gamma_c})$;

(iii) $\mathrm{q}_{1,\Sigma}(\mathfrak{f}, c^{\pm}\mathfrak{f}) = \mathrm{i}(\mathfrak{f}, G_{1,\Sigma}c^{\pm}\mathfrak{f})_{V_{\rho_1}} \geq 0 \quad \forall \mathfrak{f} \in \ker(K_\Sigma^\dagger|_{\Gamma_c})$.

(iv) $WF'(U_1 c^{\pm}) \subset (\mathcal{N}^{\pm} \cup F) \times T^*\Sigma$ for $F \subset T^*M$

Then $\Lambda^{\pm}([s],[t]) := (s, \lambda^{\pm} t)_{V_1}$ where $\lambda^{\pm} := (\rho_1 G_1)^* \mathrm{i} G_{1,\Sigma} c^{\pm}(\rho_1 G_1)$

are pseudo-covariances for a quasifree Hadamard state $\omega : \mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \mathrm{q}_1) \to \mathbb{C}$.

Difficulties:

- the fiber metric on V_{ρ_1} may in general be not positive definite $\Longrightarrow$ the positivity (iii) is difficult to achieve
- pseudodifferential calculus works nice with the Hadamard condition (iv), but interact badly with gauge invariance (i) and positivity (iii)

PROPOSAL: we fix completely the gauge degrees of freedom and we construct $c^{\pm}$ modifying the method of Gérard and Wrochna for linearized Yang-Mills

OUTLINE

(I) The Cauchy radiation gauge

(II) Hodge-decomposable forms

(III) The Complete Gauge Fixing and the Phase space

(IV) Hodge-decomposable data in the Cauchy radiation gauge

(V) Hadamard states in the Cauchy radiation gauge

Joint project with Gabriel Schmid, Ph.D. student in Genoa

Maxwell's theory as a gauge theory:

(I) $V_0 = \Big(M \times \mathbb{C}, (\cdot\,,\cdot)_{V_0}\Big)$ and $V_1 = \Big(T^*M \otimes_{\mathbb{R}} \mathbb{C}, (\cdot\,,\cdot)_{V_1}\Big)$ where

$$(\cdot,\cdot)_{V_1} := \int_M g^{-1}(\bar{\cdot},\cdot)\,\mathrm{vol}_g$$

(II) set $P =: \delta d$, $K = d$ and $K^* = \delta \Longrightarrow D_1 := \delta d + d\delta$ and $D_0 = \delta d$

Because $\ker P$ is invariant under conformal rescaling we can set

$$M = \mathbb{R} \times \Sigma \qquad g = -dt^2 + h_t\,.$$

DEFINITION: $A = A_0 dt + A_\Sigma$ satisfies **Cauchy radiation gauge** on a Σ if

$$\delta A = 0 \textit{ (Lorenz gauge)} \quad \text{and} \quad A_0|_\Sigma = \partial_t A_0|_\Sigma = 0$$

REMARK: On ultrastatic spacetimes, the following gauge are equivalent:

(i) A satisfies the *Cauchy radiation gauge*;

(ii) A satisfies the *temporal gauge* $A_0 = 0$ and the *Coulomb gauge* $\delta_\Sigma A_\Sigma = 0$;

(iii) A satisfies the *radiation gauge*, *i.e. temporal gauge* and *Lorenz gauge*.

(iv) The fiber metric g^{-1} reduces to h^{-1} in the Cauchy radiation gauge

$$g^{-1}(A,A) = -(A_0,A_0) + h^{-1}(A_\Sigma,A_\Sigma) = h^{-1}(A_\Sigma,A_\Sigma) \geq 0$$

When can be achieved the Cauchy radiation gauge?

DEFINITION: We call **space of Hodge-decomposable k-forms**

$$\Omega^k_{\mathrm{H}}(\Sigma) := \Big(\big(\mathrm{ran}(\mathrm{d}_\Sigma) \cap \Omega^k_c(\Sigma)\big) \oplus \ker(\delta_\Sigma)\Big) \otimes_{\mathbb{R}} \mathbb{C}$$

We call **space of radiation k-forms** $\Omega^k_{\mathrm{R}}(\mathrm{M})$

$$\Omega^k_{\mathrm{R}}(\mathrm{M}) := \begin{cases} \{\omega \in \Omega^k_{\mathrm{sc}}(\mathrm{M}) \,|\, \omega_\Sigma|_\Sigma \in \Omega^k_{\mathrm{H},c}(\Sigma)\} & \text{for } k > 0 \\ C^\infty_{\mathrm{sc}}(\mathrm{M}) & \text{for } k = 0 \end{cases},$$

where $\Omega^k_{\mathrm{H},c}(\Sigma) := \Omega^k_{\mathrm{H}}(\Sigma) \cap \Omega^k_c(\Sigma)$ and $\omega_\Sigma = \omega - dt \wedge (\partial_t \lrcorner \omega)$.

REMARKS: On a Riemannian manifold (Σ, h) it holds:

(i) $\Omega^0_{\mathrm{H}}(\Sigma) = C^\infty_c(\Sigma, \mathbb{C})$ and $\Omega^1_{\mathrm{H}}(\Sigma) = \Big(\mathrm{ran}(\mathrm{d}_\Sigma|_{\Omega^1_c}) \oplus \ker(\delta_\Sigma)\Big) \otimes_{\mathbb{R}} \mathbb{C}$;

(ii) If Σ is compact, then $\Omega^k_{\mathrm{H}}(\Sigma) = \Omega^k(\Sigma) \otimes_{\mathbb{R}} \mathbb{C}$ (by Hodge-decomposition);

(iii) the Hodge-Laplacian Δ_0 is *invertible* modulo a constant, namely

$$\Delta_0 \colon \frac{\Omega^0_{\mathrm{H}}(\Sigma)}{\{f \in \Omega^0_{\mathrm{H}}(\Sigma) \,|\, f \text{ is constant}\}} \to \mathrm{ran}(\delta_\Sigma|_{\Omega^1_{\mathrm{H}}(\Sigma)}) \text{ is bijective.}$$

PROPOSITION: $(\mathrm{M} = \mathbb{R} \times \Sigma, g = -dt^2 + h_t)$ be globally hyperbolic
For any $A \in \Omega^1_{\mathrm{R}}(\mathrm{M})$, there exists a $f \in \Omega^0_{\mathrm{R}}(\mathrm{M})$ (unique up to a constant) such that $A' := A + \mathrm{d}f$ satisfies the Cauchy radiation gauge.

Sketch of the proof: -decompose $A = A_0 \mathrm{d}t + A_\Sigma$
- A' satisfies the Cauchy radiation gauge if we can solve the system

$$\begin{cases} \mathrm{D}_0 f & = -\delta A \\ \partial_t f|_\Sigma & = -A_0|_\Sigma \\ \Delta_0 f|_\Sigma & = -\delta_\Sigma A_\Sigma|_\Sigma \end{cases}$$

- the Hodge-Laplacian Δ_0 is invertible (modulo a constant), so che Cauchy problem for f is well-posed. □

REMARK: f is unique (up to a constant), so the gauge is fixed completely, *i.e.*

$$\frac{\ker(\mathrm{P}|_{\Gamma_{\mathrm{sc}}})}{\mathrm{ran}(\mathrm{K}|_{\Gamma_{\mathrm{sc}}})} \simeq \ker(\mathrm{D}_1|_{\Gamma_{\mathrm{sc}}}) \cap \ker(\mathrm{K}^*|_{\Gamma_{\mathrm{sc}}}) \cap \ker(\mathrm{R}|_{\Gamma_{\mathrm{sc}}})$$

where $\mathrm{R} = U_1 \mathrm{R}_\Sigma \rho_1$ and $\mathrm{R}_\Sigma(a_0, \pi_0, a_\Sigma, \pi_\Sigma) := (a_0, \pi_0, 0, 0)$

PROPOSITION (phase space): The following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{V}_{\mathrm{P}} := \dfrac{\ker(\mathsf{K}^*|_{\Gamma_{\mathbf{G}}})}{\mathrm{ran}(\mathsf{P}|_{\Gamma_{\mathbf{G}}})} & \xrightarrow{\quad [\mathsf{G}_1] \quad} & \dfrac{\ker(\mathsf{P}|_{\Gamma_{\mathbf{R}}})}{\mathrm{ran}(\mathsf{K}|_{\Gamma_{\mathbf{R}}})} \\
\Big\downarrow [\rho_1 \mathsf{G}_1] & \searrow^{[\mathsf{G}_1]} & \Big\uparrow \\
\mathcal{V}_{\Sigma} := \dfrac{\ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_{\mathbf{H}}})}{\mathrm{ran}(\mathsf{K}_\Sigma|_{\Gamma_{\mathbf{H}}})} & \xrightarrow{\quad [U_1] \quad} & \dfrac{\ker(\mathsf{D}_1|_{\Gamma_{\mathbf{R}}}) \cap \ker(\mathsf{K}^*|_{\Gamma_{\mathbf{R}}})}{\mathsf{K}(\ker(\mathsf{D}_0|_{\Gamma_{\mathbf{R}}}))} \\
\Big\downarrow \mathsf{T}_\Sigma & & \Big\uparrow \\
\mathcal{V}_{\mathrm{R}} := \ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_{\mathbf{H}}}) \cap \ker(\mathsf{R}_\Sigma|_{\Gamma_{\mathbf{H}}}) & \xrightarrow{\ U_1\ } & \ker(\mathsf{D}_1|_{\Gamma_{\mathbf{R}}}) \cap \ker(\mathsf{K}^*|_{\Gamma_{\mathbf{R}}}) \cap \ker(\mathsf{R}|_{\Gamma_{\mathbf{R}}})
\end{array}
$$

where $\Gamma_{\mathsf{G}}(\mathsf{V}_1) := \{A \in \Gamma_c(\mathsf{V}_1) \mid \mathsf{G}_1 A \in \Gamma_{\mathsf{R}}(\mathsf{V}_1)\} \subset \Gamma_c(\mathsf{V}_1)$

and $\Gamma_{\mathsf{H}}(\mathsf{V}_{\rho_1}) := (C_c^\infty(\Sigma) \otimes_{\mathbb{R}} \mathbb{C})^2 \oplus (\Omega^1_{\mathsf{H},c}(\Sigma) \otimes_{\mathbb{R}} \mathbb{C})^2$.

DEFINITION: **space of Hodge-decomp data in the Cauchy radiation gauge**

$$\mathcal{V}_{\mathrm{R}} := \ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_{\mathbf{H}}}) \cap \ker(\mathsf{R}_\Sigma|_{\Gamma_{\mathbf{H}}}),$$

We conclude the classical theory, by endowing $\mathcal{V}_{\mathrm{R}}$ with an Hermitian form $q_{\Sigma,\mathrm{R}}$

- Decomposing $A = A_0 dt + A_\Sigma$, we set

$$\rho_0 \colon f \mapsto \begin{pmatrix} f|_\Sigma \\ \frac{1}{\mathrm{i}}\partial_t f|_\Sigma \end{pmatrix} \quad \text{and} \quad \rho_1 \colon A \mapsto \begin{pmatrix} A_0|_\Sigma \\ \frac{1}{\mathrm{i}}\partial_t A_0|_\Sigma \\ A_\Sigma|_\Sigma \\ \frac{1}{\mathrm{i}}\partial_t A_\Sigma|_\Sigma \end{pmatrix}$$

- By construction $[\rho_1 \mathrm{G}_1] \colon (\mathcal{V}_{\mathrm{P}}, q_1) \to (\mathcal{V}_\Sigma, q_{1,\Sigma})$ is an unitary isomorphism

$$q_{1,\Sigma}([\cdot],[\cdot]) = \mathrm{i}([\cdot], \mathrm{G}_{1,\Sigma}[\cdot])_{\mathrm{V}_{\rho_1}} \qquad \mathrm{G}_{1,\Sigma} = \frac{1}{\mathrm{i}} \begin{pmatrix} 0 & -\mathbb{1} & 0 & 0 \\ -\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}$$

- We define $\mathrm{q}_{\Sigma,\mathrm{R}}$ such that $\mathrm{T}_\Sigma : (\mathcal{V}_\Sigma, \mathrm{q}_{1,\Sigma}) \to (\mathcal{V}_{\mathrm{R}}, \mathrm{q}_{\Sigma,\mathrm{R}})$ is unitary

$$\mathrm{q}_{\Sigma,\mathrm{R}}(\cdot,\cdot) = \mathrm{i}(\cdot \mathrm{G}_{\Sigma,\mathrm{R}} \cdot)_{\mathrm{V}_{\rho_1}} \qquad \mathrm{G}_{\Sigma,\mathrm{R}} = \frac{1}{\mathrm{i}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}$$

Summing up: unitary isomorphisms $(\mathcal{V}_{\mathrm{P}}, q_1) \simeq (\mathcal{V}_\Sigma, q_{1,\Sigma}) \simeq (\mathcal{V}_{\mathrm{R}}, q_{\Sigma,\mathrm{R}})$

Next goal: Write the $\mathsf{T}_\Sigma \colon \mathcal{V}_\Sigma \to \mathcal{V}_{\mathrm{R}}$ more explicitly

$$\mathcal{V}_{\mathrm{R}} := \ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_{\mathbf{H}}}) \cap \ker(\mathsf{R}_\Sigma|_{\Gamma_{\mathbf{H}}}) = \mathrm{ran}(\mathsf{T}_\Sigma) \qquad \ker(\mathsf{T}_\Sigma) = \mathrm{ran}(\mathsf{K}_\Sigma|_{\Gamma_{\mathbf{H}}})$$

To compute T_Σ we follows this ansatz

$$T_\Sigma = \mathbb{1} - \mathsf{K}_\Sigma(\mathsf{R}_\Sigma\mathsf{K}_\Sigma)^{-1}\mathsf{R}_\Sigma$$

PROPOSITION: Let (Σ, h) be a Riemannian manifold and $\pi_\delta := \mathbb{1} - \mathrm{d}_\Sigma\Delta_0^{-1}\delta_\Sigma$. There exists a map $\mathsf{T}_\Sigma \colon \Gamma_{\mathsf{H}}(\mathsf{V}_{\rho_1}) \to \Gamma_{\mathsf{H}}(\mathsf{V}_{\rho_1})$ defined by

$$T_\Sigma = \begin{pmatrix} 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{pmatrix} \pi_\delta & 0 \\ 0 & \pi_\delta \end{pmatrix} \end{pmatrix}$$

satisfies the following properties

(i) $T_\Sigma = \mathbb{1} - \mathsf{K}_\Sigma(\mathsf{R}_\Sigma\mathsf{K}_\Sigma)^{-1}\mathsf{R}_\Sigma$ on $\ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_{\mathbf{H}}})$

(ii) $\mathsf{T}_\Sigma^2 = \mathsf{T}_\Sigma$ and $\mathsf{T}_\Sigma|_{\mathcal{V}_{\mathrm{R}}} = \mathbb{1}$;

(iii) $\ker(\mathsf{T}_\Sigma) = \mathrm{ran}(\mathsf{K}_\Sigma|_{\Gamma_{\mathbf{H}}})$;

(iv) $\mathrm{ran}(\mathsf{T}_\Sigma) = \ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_{\mathbf{H}}}) \cap \ker(\mathsf{R}_\Sigma|_{\Gamma_{\mathbf{H}}})$.

Finally we can construct Hadamard states!

- By the standard deformation argument, we assume

$$(\mathsf{M}, g) \quad \text{to be ultrastatic and of bounded geometry}$$

- Using pseudodifferential calculus and spectral calculus, we can construct a square root ϵ_i of the Hodge-Laplacian Δ_i satisfying

$$\epsilon_i \pi_\delta = \pi_\delta \varepsilon_i \quad \text{modulo } \mathcal{W}^{-\infty}$$

where again $\pi_\delta = \mathbb{1} - \mathrm{d}_\Sigma \Delta_0^{-1} \delta_\Sigma$

- Finally consider the pseudodifferential projectors $\pi^\pm$ defined by

$$\pi^\pm := \frac{1}{2} \begin{pmatrix} \mathbb{1} & \pm\varepsilon_0^{-1} & 0 & 0 \\ \pm\varepsilon_0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} & \pm\varepsilon_1^{-1} \\ 0 & 0 & \pm\varepsilon_1 & \mathbb{1} \end{pmatrix}$$

Notice that: since $\mathsf{D}_i = (\partial_t + \epsilon_i)(\partial_t - \epsilon_i)$ modulo $\mathcal{W}^{-\infty}$, then

Hadamard condition: $\mathrm{WF}'(U_1 \pi^\pm) \subset (N^\pm \cup F) \times \mathrm{T}^*\Sigma$ for $F = \{k = 0\} \subset T^*\mathsf{M}$

THEOREM [S.M., Schmid] The operators $c^{\pm} := \mathsf{T}_\Sigma \pi^{\pm} \mathsf{T}_\Sigma$, defined by

$$T_\Sigma = \begin{pmatrix} 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{pmatrix} \pi_\delta & 0 \\ 0 & \pi_\delta \end{pmatrix} \end{pmatrix} \qquad \pi^{\pm} := \frac{1}{2} \begin{pmatrix} \mathbb{1} & \pm\varepsilon_0^{-1} & 0 & 0 \\ \pm\varepsilon_0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} & \pm\varepsilon_1^{-1} \\ 0 & 0 & \pm\varepsilon_1 & \mathbb{1} \end{pmatrix}$$

have the following properties:

$$\begin{array}{ll} \text{(i)} & (c^{\pm})^{\dagger} = c^{\pm} \quad \text{and} \quad c^{\pm}(\mathrm{ran}(\mathsf{K}_\Sigma|_{\Gamma^\infty_{\mathbf{H}}})) \subset \mathrm{ran}(\mathsf{K}_\Sigma) \\ \text{(ii)} & (c^{+} + c^{-})\mathfrak{f} = \mathfrak{f} \mod \mathrm{ran}(\mathsf{K}_\Sigma|_{\Gamma^\infty_{\mathbf{H}}}) \quad \forall \mathfrak{f} \in \ker(\mathsf{K}^{\dagger}_\Sigma|_{\Gamma^\infty_{\mathbf{H}}}) \\ \text{(iii)} & \pm\, \mathrm{q}_{1,\Sigma}(\mathfrak{f}, c^{\pm}\mathfrak{f}) \geq 0 \quad \forall \mathfrak{f} \in \ker(\mathsf{K}^{\dagger}_\Sigma|_{\Gamma^\infty_{\mathbf{H}}}) \\ \text{(iv)} & \mathrm{WF}'(U_1 c^{\pm}) \subset (\mathcal{N}^{\pm} \cup F) \times \mathrm{T}^*\Sigma \ \text{ for } F = \{k = 0\} \subset T^*\mathrm{M} \end{array}$$

In other words,

$$\lambda^{\pm} := (\rho_1 \mathsf{G}_1)^* \lambda^{\pm}_\Sigma (\rho_1 \mathsf{G}_1) \qquad \text{where} \qquad \lambda^{\pm}_\Sigma := \pm \mathrm{i} \mathsf{G}_{1,\Sigma} c^{\pm}$$

are the pseudo-covariances of a quasi-free Hadamard state on $\mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \mathrm{q}_1)$.

Sketch of the proof

(i) Since $\varepsilon_i = \varepsilon_i^*$ are formally self-adjointw.r.t the Hodge-inner product on Σ

$$(\pi^{\pm})^{\dagger} = \mathsf{G}_{1,\Sigma}^{-1}(\pi^{\pm})^{*}\mathsf{G}_{1,\Sigma} = \pi^{\pm}\,,$$

Then $\pi^{\pm}$, T_{Σ} and also $c^{\pm}$ are formally self-adjoint w.r.t. $\sigma_{1,\Sigma}$.

(ii) $\pi^{+} + \pi^{-} = \mathbb{1}$ on $\Gamma^{\infty}_{\mathbf{H}}(\mathsf{V}_{\rho_1})$ and hence

$$(c^{+} + c^{-})\mathfrak{f} = T_{\Sigma}^{2}\mathfrak{f} = T_{\Sigma}\mathfrak{f} = \mathfrak{f} \quad \mathrm{mod} \quad \mathrm{ran}(\mathsf{K}_{\Sigma}|_{\Gamma^{\infty}_{\mathbf{H}}})$$

for all $\mathfrak{f} \in \ker(\mathsf{K}_{\Sigma}^{\dagger}|_{\Gamma^{\infty}_{\mathbf{H}}})$, where in the last step we used that T_{Σ} is a bijection between $\mathcal{V}_{\mathrm{P}}$ and $\mathcal{V}_{\Sigma}$ together with $T_{\Sigma} = \mathbb{1}$ on $\ker \mathsf{R}_{\Sigma}$.

(iii) we compute

$$\pm \mathrm{q}_{1,\Sigma}(\mathfrak{f}, c^{\pm} f) = \pm \mathrm{q}_{1,\Sigma}(\mathfrak{f}, \mathsf{T}_{\Sigma}\pi^{\pm}\mathsf{T}_{\Sigma} f) = \pm \mathrm{q}_{\Sigma,\mathsf{R}}(\mathsf{T}_{\Sigma}\mathfrak{f}, \pi^{\pm}\mathsf{T}_{\Sigma} f) \geq 0$$

(iv) follows because $\pi^{\pm}$ commutes with T_{Σ} modulo a smooth kernel and $\pi^{\pm}$ satisfies the Hadamard condition □

WHAT WE HAVE SEEN AND WHAT COMES NEXT?

MAXWELL'S THEORY:

- Gauge fixing is useful for getting positivity and gauge invariance, but the price to pay is reducing the space of classical observables if Σ not compact

- For generic manifold , we can construct Hadamard projectors $\pi^{\pm}$, but it is not clear that they commute with T_{Σ} (even modulo smoothing)

FUTURE WORK: LINEARIZED GRAVITY

- Gauge fixing completely the linearized gravity on the level of initial data: *Synchronous, de Donder, traceless-gauge, ...*

- Constructing T_{Σ} is very challenging from a technical point of view (two-tensors does not make your life nicer)

- We cannot use the deformation argument, so we need to modify $\pi^{\pm}$ such that the operators $c^{\pm} = T_{\Sigma}\pi^{\pm}\mathrm{T}_{\Sigma}$ satisfies the Hadamard conditions

THANKS for your attention!