A new construction of algebraic states for CAR algebras[1]

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[1] Joint work with F. Finster and C. Röken

Outline

- **On the algebraic approach to QFT**
- **Quasi-Free States and Fermionic Projectors**
- **Mass Oscillation Property**
- **Hadamard States**

Based on:

- F. Finster, S. M., C. Röken, arXiv:1501.05522 [math-ph]

AQFT - I: Dirac field

- **Spinor bundle** $S\mathcal{M} \simeq \mathcal{M} \times \mathbb{C}^4$ and **cospinor bundle** $S^*\mathcal{M} \simeq \mathcal{M} \times (\mathbb{C}^4)^*$, with $\mathcal{M} \simeq \mathbb{R} \times \Sigma$ 4-dim *globally hyperbolic spacetime* :

$$ds^2 = \beta^2 dt^2 - h_t; \quad \beta \in C^\infty(M; \mathbb{R}^+) \text{ and } h_t \in \text{Riem}(\Sigma); \forall t \in \mathbb{R}.$$

- Spinors $\psi \in C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4)$ and cospinors $\varphi \in C^\infty_{sc}(\mathcal{M}, (\mathbb{C}^4)^*)$.

- Dirac conjugation map: $A : \ C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \underset{(\leftarrow)}{\longrightarrow} C^\infty_{sc}(\mathcal{M}, (\mathbb{C}^4)^*)$.

- **Spin scalar product**: $\prec \cdot \,|\, \cdot \succ_x : C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \times C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \to \mathbb{C}$

$$\prec \psi | \widetilde{\psi} \succ_x := \big((A\psi)\widetilde{\psi}\big)(x).$$

- **Space-time inner product**: $<\cdot|\cdot> \ : \ C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \times C^\infty_{c}(\mathcal{M}, \mathbb{C}^4) \to \mathbb{C}$

$$<\psi|\widetilde{\psi}> = \int_{\mathcal{M}} \prec \psi | \widetilde{\psi} \succ_x d\mu_g \, .$$

AQFT - II: Dynamics

- **Dirac operator** on SM and its dual on S^*M:

$$\mathcal{D}\psi_m \doteq (i\gamma^\mu\nabla_\mu + \mathcal{B} - m)\psi_m = 0, \qquad \mathcal{D}^*\varphi_m = (-i\gamma^\mu\nabla_\mu + \mathcal{B} - m)\varphi_m = 0.$$

- **Causal propagators:** $K^{(*)} : C_c^\infty(\mathcal{M}, (\mathbb{C}^4)^{(*)}) \to C_{sc}^\infty(\mathcal{M}, (\mathbb{C}^4)^{(*)})$

$$\mathcal{D}^{(*)} \circ K^{(*)} = 0 = K^{(*)} \circ \mathcal{D}^{(*)}|_{C_c^\infty(\mathcal{M},(\mathbb{C}^4)^{(*)}}$$

$$supp(K^{(*)}(f)) \subseteq J(supp(f)), \qquad \forall f \in C_c^\infty(\mathcal{M}, (\mathbb{C}^4)^{(*)})$$

- **Hilbert spaces:**

$$\mathcal{H}_m^s := \overline{\left(\mathcal{S}ol(\mathcal{D}) \simeq \frac{C_c^\infty(\mathcal{M}, \mathbb{C}^4)}{\mathcal{D}C_c^\infty(\mathcal{M}, \mathbb{C}^4)}, (\cdot \mid \cdot)_m^s \doteq \int_\Sigma \prec \cdot |\not{\psi} \cdot \succ_x d\Sigma \right)}$$

$$\mathcal{H}_m^c := \overline{\left(\mathcal{S}ol(\mathcal{D}^*)) \simeq \frac{C_c^\infty(\mathcal{M}, (\mathbb{C}^4)^*)}{\mathcal{D}^*C_c^\infty(\mathcal{M}, \mathbb{C}^4)^*)}, (\cdot \mid \cdot)_m^c \doteq \int_\Sigma \prec A^{-1} \cdot |\not{\psi}\, A^{-1} \cdot \succ_x d\Sigma \right)}$$

AQFT - III: CAR Algebra

- unital Borchers-Uhlmann $*$-algebra: $\mathcal{A} = \bigoplus_{k=0}^{\infty} \left(\mathcal{S}ol(\mathcal{D}_m) \oplus \mathcal{S}ol(\mathcal{D}_m^*) \right)^{\otimes k}$
  - $\bullet : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined by $\{u_k\} \bullet \{v_k\} = \{w_k = \sum_{i+j=k} u_i \otimes v_j\}$
  - $\mathbb{1} = \{1, 0, 0, \ldots\}, \Phi(\psi_m) = \left\{0, \binom{\psi_m}{0}, 0, \ldots\right\}, \Psi(\varphi_m) = \left\{0, \binom{0}{\varphi_m}, 0, \ldots\right\}$
  - $*$-product: $\left\{0, 0, \binom{\psi_m}{\varphi_m} \otimes \binom{\widetilde{\psi}_m}{\widetilde{\varphi}_m}, \ldots\right\}^* = \left\{0, 0, \begin{pmatrix} A^{-1}\widetilde{\varphi}_m \\ A\widetilde{\psi}_m \end{pmatrix} \otimes \begin{pmatrix} A^{-1}\varphi_m \\ A\psi_m \end{pmatrix}, \ldots\right\}$

- We encode the CARs in the ideal $\mathcal{I} \subset \mathcal{A}$:
  - $\Phi(\psi_m) \otimes \Phi(\widetilde{\psi}_m) + \Phi(\widetilde{\psi}_m) \otimes \Phi(\psi_m)$
  - $\Psi(\varphi_m) \otimes \Psi(\widetilde{\varphi}_m) + \Psi(\widetilde{\varphi}_m) \otimes \Psi(\varphi_m)$
  - $\Psi(\varphi_m) \otimes \Phi(\psi_m) + \Phi(\psi_m) \otimes \Psi(\varphi_m) - \left(A^{-1}\varphi_m \mid \psi_m\right)_m^s \mathbb{1}$

- **Algebra of fields:** $\mathcal{F} \doteq \dfrac{\mathcal{A}}{\mathcal{I}}$

AQFT - IV: States

- **Algebraic state** $\omega : \mathcal{F} \to \mathbb{C}$ such that: $\omega(\mathbb{1}) = 1, \qquad \omega(h^* h) \geq 0, \quad \forall\, h \in \mathcal{F}.$

N.B.: Choosing a state ω is equivalent to assigning $\omega_n(h_1, \ldots, h_n)$ $\forall n \in \mathbb{N}$ and $\forall h_i \in \mathcal{F}$.

- **Quasi-free states:** $\qquad \omega_{2n+1}(h_1, \ldots, h_{2n+1}) = 0$

$$\omega_{2n}(h_1, \ldots, h_{2n}) = \sum_{\sigma \in S'_{2n}} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^{n} \omega_2\big(h_{\sigma(2i-1)}, h_{\sigma(2i)}\big).$$

Question: *Are all states physically acceptable?*

Of course not! Minimal requirements are:

i) covariant construction of Wick polynomials to deal with interactions,

ii) same UV behaviour of the Minkowski vacuum,

iii) finite quantum fluctuations of all observables.

Answer: **Hadamard States**

Microlocal Analysis

- **Space of symbols**

$$S^\lambda := \Big\{ q \in C^\infty(U \subset \mathbb{R}^n \times \mathbb{R}^n) \,:\, |\, D_\xi^\alpha\, q(x,\xi)\,| \leq C_{\alpha,K}\,(1 + |\,\xi\,|)^{\lambda - |\alpha|}$$
$$\forall \alpha \in \mathbb{N}^n,\ \forall K \subset \mathbb{R}^n \text{ compact and } x \in K,\ \xi \in \mathbb{R}^n,\ C_{\alpha,K} \in \mathbb{R} \Big\}.$$

- **Pseudo-differential operators** $Q : C_c^\infty(U) \to C^\infty(U)$ such that

$$Qu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)\, d\xi \quad u \in C_c^\infty(U), \ \ q \in S^\lambda.$$

- **Wavefront set** for $u \in D'(U)$:

$$\mathrm{WF}(u) := \bigcap_{Qu \in C^\infty(U)} \Big\{ (x,\xi) \in U \times \mathbb{R}^n \setminus 0 : q(x,\xi) = 0 \Big\}.$$

- If $\Xi \in C^\infty(V,U)$ then $\mathrm{WF}(\Xi^* u) = \Xi^* \mathrm{WF}(u) := \Big\{ (\Xi^* x, \Xi^* \xi) : (x,\xi) \in \mathrm{WF}(u) \Big\}$.
- We extend the definition of WF on SM to be $\mathrm{WF}(u) := \bigcup_i \mathrm{WF}(u_i)$.

AQFT - V: Hadamard States

- A (quasi-free) state ω satisfies the **Hadamard condition** if and only if

$$WF(\omega_2) = \left\{(x, y, \xi_x, \xi_y) \in T^*M^{\otimes 2} \setminus 0 \,\middle|\, (x, \xi_x) \sim (y, -\xi_y), \quad \xi_x \triangleright 0\right\},$$

where $(x, \xi_x) \sim (y, -\xi_y)$ means that x and y are connected by a null geodesic and $-\xi_y$ is the parallel transport of the co-parallel co-vector ξ_x; whereas $\xi_x \triangleright 0$ implies that ξ_x is future-pointing.

Question: *How many Hadamard states do we know?*

- deformation arguments (existence)
- static spacetime (H. Sahlmann and R. Verch)
- highly symmetric spacetimes, e.g., de Sitter spacetime
- holographic techniques (C. Dappiaggi, V. Moretti and N. Pinamonti)
- pseudodifferential calculus (C. Gérard and M. Wrochna)

Question: *Does there exist an explicit method that does not use the symmetries?*

Quasi-Free States and Projection Operators

Lemma 3.3 (*H. Araki: On quasifree states of CAR and Bogoliubov automorphisms. (1970/71).*)

Let R be a bounded symmetric operator on $(\mathscr{H}^s_m, (.|.)^s_m)$ with the following properties

(a) $R + ARA = \mathbb{1}$,

(b) $0 \leq R = R^* \leq \mathbb{1}$.

Then there exists a unique quasi-free state ω on $\mathcal{F}$ such that

$$\omega_2\Big(\Psi(\varphi_m)\Phi(\psi_m)\Big) = \big(A^{-1}\varphi_m \mid R\psi_m\big)^s_m$$

- Our motto will be $\Longrightarrow$ "Split the $\mathscr{H}$ilbert space!"

(...but how?)

The Fermionic Projector in a Strip of Space-Time

[F. Finster and M. Reintjes: A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds I – Space-times of finite lifetime, arXiv:1301.5420 [math-ph], to appear in Adv. Theor. Math. Phys. (2015).]

- Begin with a simple example, to explain the basic idea:

$$\Omega \subset (-T, T) \times \Sigma \subset \mathcal{M}$$

- As before the scalar product: $(\cdot|\cdot)_m : C^\infty_{sc}(\Omega, S\,\Omega) \times C^\infty_{sc}(\Omega, S\,\Omega) \to \mathbb{C}$

$$(\varphi_m|\psi_m)_m := 2\pi \int_\Sigma \prec\psi_m|\not{\nu}\varphi_m\succ_x \, d\Sigma.$$

- $\mathscr{H}_m := \left(C^\infty_{sc}(\Omega, S\,\Omega), (\cdot|\cdot)_m\right)$ is an Hilbert space.

- The space-time inner product: $<\cdot|\cdot> \; : \; C^\infty_{sc}(\Omega, S\,\Omega) \times C^\infty_{sc}(\Omega, S\,\Omega) \to \mathbb{C}$

$$<\psi_m|\varphi_m> = \int_\Omega \prec\psi_m|\varphi_m\succ_x \, d\mu_\Omega \qquad \textit{(well defined)}$$

$$|<\varphi_m|\psi_m>| \leq c \, \|\varphi_m\|_m \, \|\psi_m\|_m \qquad \textit{(bounded)}.$$

- **Riesz representation theorem**

$$<\varphi_m|\psi_m> = (\varphi_m \,|\, \mathcal{S}\,\psi_m)_m.$$

- From the spectral theorem, we can construct an orthonormal projector

$$\chi^{\pm}(\mathcal{S}) = \int_{\sigma} \chi^{\pm}(\lambda)\, dE_{\lambda}.$$

- Finally we can obtain a quasi-free state in the whole space-time as:

$$\mathcal{P} := \chi^{+}(\mathcal{S}) \cdot \mathcal{K}.$$

Problems:

- For $T \to \infty$ the space-time inner product is not well defined,
- $\mathcal{P}(x, y)$ is in general not Hadamard!
  [*C. Fewster and B. Lang: Pure quasifree states of the Dirac field from the fermionic projector, arXiv:1408.1645 [math-ph].*]

Mass Oscillation Property

[F. Finster and M. Reintjes: A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds II – Space-times of infinite lifetime, arXiv:1312.7209 [math-ph].]

- Families of solutions of families of Dirac equations:

$$\Psi := (\psi_m)_{m \in \mathrm{I}=(m_L, m_R)} \in \mathscr{H}^\infty .$$

- New **scalar product**: $(\cdot\,,\cdot) : \mathscr{H}^\infty \times \mathscr{H}^\infty \to \mathbb{C}$

$$(\Psi|\Phi) = \int_{\mathrm{I}} (\psi_m|\varphi_m)_m \, dm.$$

- Integration over mass as the operator

$$\mathfrak{p} \;:\; \mathscr{H}^\infty \to C^\infty_{\mathrm{sc}}(\mathcal{M}, S\mathcal{M})\,, \quad \mathfrak{p}\Psi = \int_I \psi_m \, dm \,.$$

- "$\mathfrak{p}$" **space-time inner product:**

$$<\mathfrak{p}\,\cdot\,|\mathfrak{p}\,\cdot> \;:\; \mathscr{H}^\infty \times \mathscr{H}^\infty \to \mathbb{C}$$

Q: *Is the new space-time inner product bounded?*

- The Dirac operator has the **strong mass oscillation property** in $I = (m_L, m_R)$ if there exists a constant $c > 0$ such that

$$\left| <\mathfrak{p}\Psi \mid \mathfrak{p}\Phi> \right| = \left| \int_{\mathbb{R}^4} \prec \mathfrak{p}\Psi \,|\mathfrak{p}\Phi \succ_x d^4x \right| \leq c \int_I \|\psi_m\|_m \, \|\varphi_m\|_m \, dm$$

for all families of solutions $\Psi, \Phi \in \mathscr{H}^\infty$.

- Then there exists a family of linear operators $(\mathcal{S}_m)_{m \in I}$ with $\mathcal{S}_m \in \mathrm{L}(\mathscr{H}_m)$ which are uniformly bounded $\sup_{m \in I} \|\mathcal{S}_m\| < \infty$, such that

$$<\mathfrak{p}\Psi|\mathfrak{p}\Phi> = \int_I (\psi_m \,|\, \mathcal{S}_m \, \varphi_m)_m \, dm \qquad \text{for all } \Psi, \Phi \in \mathscr{H}^\infty \, .$$

- The operator $\mathcal{S}_m$ is **uniquely determined** for every $\boldsymbol{m} \in I$ by demanding that for all $\Psi, \Phi \in \mathscr{H}^\infty$, the functions $(\psi_m|\mathcal{S}_m\varphi_m)_m$ are continuous in m.

Q: *Where does the strong mass oscillation property hold?*

Q: *Does Fermionic Projector satisfy the Hadamard condition?*

The Fermionic Projector in the Minkowski Vacuum $(\mathcal{B} = 0)$

- Family of solutions of the Dirac equations $\Psi = (\psi_m)_{m\in I} \in \mathscr{H}^\infty$ in momentum space via the Fourier transform of a solution :

$$\psi_m(\xi) = 2\pi\, \delta(\xi^2 - m^2)\, sign(\xi^0)\, (\not{\xi} + m)\, \gamma^0 \hat{\psi}^0_m(\vec{\xi})\,.$$

- After integration over m

$$(\mathfrak{p}\Psi)(\xi) = 2\pi\, \chi_I(m)\, \frac{1}{2m}\, sign(\xi^0)\, (\not{\xi} + m)\, \gamma^0 \hat{\psi}^0_m(\vec{\xi})\Big|_{m=\sqrt{\xi^2}}$$

we compute the "$\mathfrak{p}$" space-time inner product

$$<\mathfrak{p}\psi|\mathfrak{p}\varphi> = \int_{\mathbb{R}^4} \frac{d^4\xi}{4\pi^2}\, \chi_I(m)\, \frac{1}{2m} \prec \gamma^0 \hat{\psi}^0_m(\vec{\xi})\,|\, (\not{\xi} + m)\, \gamma^0 \hat{\varphi}^0_m(\vec{\xi}) \succ \Big|_{m=\sqrt{\xi^2}}\,.$$

- Reparametrizing the ξ^0-integral as an integral over m, we obtain

$$<\mathfrak{p}\psi|\mathfrak{p}\varphi> = \frac{1}{4\pi^2} \int_I dm \int_{\mathbb{R}^3} \frac{d^3\xi}{2\,|\xi^0|} \prec \gamma^0 \hat{\psi}^0_m(\vec{\xi})\,|\, (\not{\xi} + m)\, \gamma^0 \hat{\varphi}^0_m(\vec{\xi}) \succ \Big|_{\xi^0=\pm\sqrt{|\vec{\xi}|^2+m^2}}\,.$$

- Using the Schwarz inequality and applying Plancherel's theorem

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\varphi{>}| \leq \frac{1}{4\pi^2}\int_I dm \int_{\mathbb{R}^3} \|\hat{\psi}^0_m(\vec{\xi})\| \, \|\hat{\varphi}^0_m(\vec{\xi})\| \, d^3\xi \leq 2\pi \int_I \|\psi_m\|_m \, \|\varphi_m\|_m \, dm \,.$$

- Then exists an operator $\mathcal{S}_m$ uniquely determined for every $m \in I$:

$$\mathcal{S}_m(\vec{\xi}) := \sum_{\xi^0 = \pm\omega(\vec{\xi})} \frac{\not{\xi} + m}{2\,\xi^0(\vec{\xi})}\,\gamma^0 \,.$$

- Applying the spectral theorem:

$$\chi^+(\mathcal{S}_m) = \Theta(\xi^0).$$

- Then the **fermonic projector** in momentum space is

$$\mathcal{P} = \chi^+(\mathcal{S}_m)\mathcal{K}_m = \underbrace{\Theta(\xi^0)}_{\text{Select the positive frequencies}} \cdot\, 2\pi\,\delta(\xi^2 - m^2)\,sign(\xi^0)\,(\not{\xi} + m)\,\gamma^0$$

External Potential in Minkowski space-time

- If the external potential $\mathcal{B}$ satisfies the conditions

$$|\mathcal{B}(t)|_{C^2} \leq \frac{c}{1+|t|^{2+\varepsilon}}$$

  then the strong mass oscillation property holds.

- For every $\Psi, \Phi \in \mathscr{H}^\infty$,

$$<\mathfrak{p}\Psi|\mathfrak{p}\Phi> = \int_I (\psi_m \,|\, \tilde{\mathcal{S}}_m \, \varphi_m)_m \, dm \,,$$

  where $\tilde{\mathcal{S}}_m : \mathscr{H}_m \to \mathscr{H}_m$ are bounded linear operators which act on the wave functions at time t_0 by

$$\tilde{\mathcal{S}}_m = \mathcal{S}_m + \frac{i}{2}\int_{-\infty}^{\infty} \varepsilon(t-t_0)\left[\mathcal{S}_m \, U_m^{t_0,t}\, \gamma^0 \mathcal{B}(t)\, \tilde{U}_m^{t,t_0} - \tilde{U}_m^{t_0,t}\, \gamma^0 \mathcal{B}(t)\, \mathcal{S}_m \, U_m^{t,t_0}\right] dt$$
$$- \frac{1}{2}\left(\int_{t_0}^{\infty}\int_{t_0}^{\infty} + \int_{-\infty}^{t_0}\int_{-\infty}^{t_0}\right) \tilde{U}_m^{t_0,t}\, \gamma^0 \mathcal{B}(t)\, \mathcal{S}_m \, U_m^{t,t'}\, \gamma^0 \mathcal{B}(t')\, \tilde{U}_m^{t',t_0} \, dt\, dt'$$

Using PDE methods, we worked hard!

(This is the technical core of the paper)

- We decompose $\tilde{\mathcal{S}}_m$ with respect to the above frequency splitting,

$$\tilde{\mathcal{S}}_m = \tilde{\mathcal{S}}^{\mathrm{D}} + \Delta\tilde{\mathcal{S}}\,, \qquad \text{where} \qquad \tilde{\mathcal{S}}^{\mathrm{D}} := \tilde{\mathcal{S}}^+_+ + \tilde{\mathcal{S}}^-_- \quad \text{and} \quad \Delta\tilde{\mathcal{S}} := \tilde{\mathcal{S}}^+_- + \tilde{\mathcal{S}}^-_+\,.$$

- Under the assumption

$$\int_{-\infty}^{\infty} |\mathcal{B}(t)|_{C^0}\, dt < \sqrt{2} - 1$$

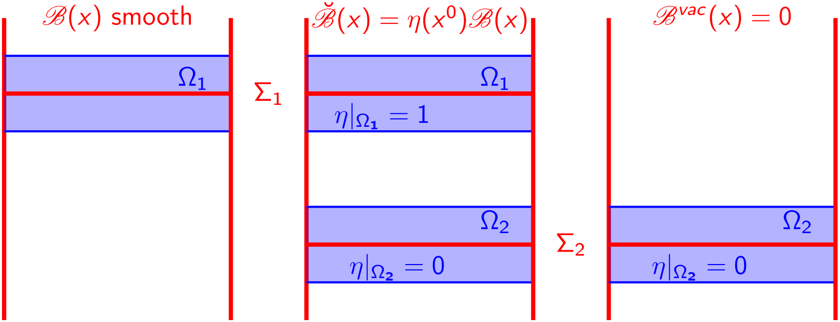
the operators $\chi^{\pm}(\tilde{\mathcal{S}}_m)$ have the representations

$$\chi^{\pm}(\tilde{\mathcal{S}}_m) = \chi^{\pm}(\mathcal{S}) + \underbrace{\frac{1}{2\pi i}\oint_{\partial B_{\frac{1}{2}}(\pm 1)} (\tilde{\mathcal{S}}_m - \lambda)^{-1}\, \Delta\tilde{\mathcal{S}}\, (\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1}\, d\lambda}_{\text{integral operator with smooth kernel}}\,,$$

- The fermionic projector is given by

$$\mathcal{P} = \chi^{+}(\tilde{\mathcal{S}}_m)\,\tilde{\mathcal{K}}_m = \chi^{+}(H)\,\tilde{\mathcal{K}}_m + \textit{ smooth contribution.}$$

Hadamard States: Three Different External Potentials



$\mathcal{P} = \chi^+(H)\,\tilde{\mathcal{K}}_m + \text{(smooth)}$ $\quad\breve{\mathcal{P}} = \chi^+(H)\,\breve{\mathcal{K}}_m + \text{(smooth)}$ $\quad\mathcal{P}^{\text{vac}} = \chi^+(H)\,\mathcal{K}_m$

- $\mathscr{P}^{\text{vac}}(x,y)$ is Hadamard in Ω_0 and then in the whole spacetime.
- $\breve{\mathscr{P}}(x,y) - \mathscr{P}^{\text{vac}}(x,y) \in C^\infty(\mathbb{R}^4)$ for all $x, y \in \Omega_0$, since $\breve{k}_m(x,y) \equiv k_m(x,y)$.
- $\breve{\mathscr{P}}(x,y)$ is Hadamard in Ω_0 and then in the whole spacetime.
- $\mathscr{P}(x,y) - \breve{\mathscr{P}}(x,y) \in C^\infty(\mathbb{R}^4)$ for all $x, y \in \Omega_1$, since $\tilde{k}_m(x,y) \equiv \breve{k}_m(x,y)$.
- $\mathscr{P}(x,y)$ is **Hadamard** in Ω_1 and then in the whole spacetime.

Conclusions

What we know:

- To every projector it is possible to associate an algebraic state.
- The construction holds also in the presence of an external potential.
- The state is Hadamard and when $\mathcal{B} = 0$ is the Poincaré vacuum.

Benefit:

- This technique works without symmetries.
- It allows to construct states for massive particles.

Future investigations:

- in which space-time the Dirac operator satisfies the mass oscillation property,
- if the states constructed turn out to be Hadamard.