Is there a natural state for Abelian Chern-Simons theory?

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Motivations and goals of my talk

" *What is a QFT?* " $\xrightarrow{\text{for a deeper understanding}}$ mathematical axioms for QFT

- Algebraic QFT :
  - ✓ $\mathcal{M} \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{M}) := \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$
    - Isotony: if $\mathcal{O} \subseteq O' \implies \mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}(\mathcal{O}')$
    - Causality: if $\mathcal{O} \cap J(\mathcal{O}') = \emptyset \implies [\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}')] = 0$
    - Covariance: isometry ι of M, $\implies \alpha_\iota \in \mathrm{Aut}(\mathcal{A})$ s.t. $\alpha_\iota \mathcal{A}(\mathcal{O}) = \mathcal{A}(\iota(\mathcal{O}))$
  - ✓ $\omega : \mathcal{A}(\mathcal{M}) \to \mathbb{C}$ s.t. normalised and positive $\Rightarrow$ GNS-theorem $(\mathscr{H}, \pi, \Omega)$
- Locally covariant QFT
  - ✓ $\mathfrak{A}$: Loc $\to$ Alg
    - Locality: $f : \mathcal{M} \to \mathcal{M}' \implies \mathfrak{f} : \mathcal{A}(\mathcal{M}) \to \mathcal{A}(\mathcal{M}')$ injective
    - Causality: $\mathcal{M}_1 \xrightarrow{f_1} \mathcal{M} \xleftarrow{f_2} \mathcal{M}_2$ caus. disjoint $\implies [\mathfrak{f}_1 \mathcal{A}(\mathcal{M}_1), \mathfrak{f}_2 \mathcal{A}(\mathcal{M}_2)] = 0$
    - Time-slice: $f : \mathcal{M} \to \mathcal{M}'$ s.t. $f(\mathcal{M}) \supset \Sigma' \implies \mathfrak{f}$ is isomorphism

  ¿ natural state ? $f : \mathcal{M} \to \mathcal{M}'$ and ω_M, $\omega_{M'}$ invariant $\omega_{M'} \circ \mathfrak{f} = \omega_M$

GOAL: Does a natural state exists for a Topological QFT?

Outline of the Talk

- **Abelian Chern-Simons theory**
- **Quantization in the algebraic approach**
- **Invariant functionals on compact surfaces**
- **¿ Natural states ?**

Based on:

- C. Dappiaggi, S.M., A. Schenkel, (arXiv:1612.04080)

The moduli space of flat $U(1)$-connections

- We consider $\mathcal{M} \simeq \mathbb{R} \times \Sigma$ without boundary and with $\dim \Sigma = 2$
- The action of Abelian Chern–Simons theory

$$S = \frac{1}{4\pi} \int_M \operatorname{tr}(A \wedge dA) \quad \Rightarrow \quad 0 = \frac{\delta S}{\delta A} = \frac{1}{2\pi} dA$$

- The moduli space of flat $U(1)$-connection:

$$\mathrm{Flat}_{U(1)} := \frac{\Omega^1_d(M)}{\Omega^1_{\mathbb{Z}}(M)} \simeq \frac{H^1(M;\mathbb{R})}{H^1(M;\mathbb{Z})} \simeq \frac{H^1(\Sigma;\mathbb{R})}{H^1(\Sigma;\mathbb{Z})} \simeq \frac{\Omega^1_d(\Sigma)}{\Omega^1_{\mathbb{Z}}(\Sigma)}$$

- In the categorical language, the assignment of the moduli spaces is a functor:

$$\mathrm{Flat}_{U(1)} : \mathsf{Man}_2^{\mathrm{op}} \to \mathsf{Ab}$$

$$\mathrm{Flat}_{U(1)}(f) := f^* : \frac{\Omega^1_d(\Sigma')}{\Omega^1_{\mathbb{Z}}(\Sigma')} \to \frac{\Omega^1_d(\Sigma)}{\Omega^1_{\mathbb{Z}}(\Sigma)} \qquad [A'] \mapsto [f^* A']$$

  - Obj(Man_2)={2-dimensional oriented manifolds}
  - Hom(Man_2)={orientation preserving open embeddings}

Observable for Abelian Chern-Simons theory

- As basic osservables we take all group characters $:= \mathrm{Hom}\big(\mathrm{Flat}_{U(1)}(\Sigma), U(1)\big)$:

$$\text{given any } \varphi \in \Omega^1_c(\Sigma) \qquad A \mapsto \exp\left(2\pi\imath \int_\Sigma \varphi \wedge A\right)$$

- This character descends to the quotient if and only if $\int_\Sigma \varphi \wedge \Omega^1_{\mathbb{Z}}(\Sigma) \subseteq \mathbb{Z}$
- Since $d\Omega^0(\Sigma) \subseteq \Omega^1_{\mathbb{Z}}(\Sigma)$ is a subgroup, Stokes' lemma implies that any $\varphi \in \Omega^1_{c,d}(\Sigma)$
- Because each exact $\varphi = d\chi \in d\Omega^0_c(\Sigma)$ yields a trivial group character, then

$$\text{group character} \simeq H^1_c(\Sigma;\mathbb{Z}) := \left\{[\varphi] \in H^1_c(\Sigma;\mathbb{R}) : \int_\Sigma \varphi \wedge H^1(\Sigma) \subseteq \mathbb{Z}\right\}$$

- The assignment of the character groups is a functor

$$\mathcal{O} := \big(H^1_c(-;\mathbb{Z}), \tau\big) : \mathsf{Man}_2 \to \mathsf{PAb}$$

$$H^1_c(f;\mathbb{Z}) := f_* \ : \ H^1_c(\Sigma;\mathbb{Z}) \longrightarrow H^1_c(\Sigma';\mathbb{Z}) \ , \quad [\varphi] \longmapsto [f_*\varphi]$$

$$\tau_\Sigma : H^1_c(\Sigma;\mathbb{Z}) \times H^1_c(\Sigma;\mathbb{Z}) \longrightarrow \mathbb{R} \qquad \tau([\varphi],[\widetilde{\varphi}])_\Sigma = \int_\Sigma \varphi \wedge \widetilde{\varphi}$$

$$\tau_{\Sigma'}\big(f_*[\varphi], f_*[\widetilde{\varphi}]\big) = \int_{\Sigma'} (f_*\varphi) \wedge (f_*\widetilde{\varphi}) = \int_\Sigma \varphi \wedge (f^* f_* \widetilde{\varphi}) = \int_\Sigma \varphi \wedge \widetilde{\varphi} = \tau_\Sigma\big([\varphi],[\widetilde{\varphi}]\big) \ ,$$

Quantization of Abelian Chern-Simons theory

- Quantization is achieved composing $\mathcal{O} : \mathsf{Man}_2 \to \mathsf{PAb}$ with $\mathfrak{CCR} : \mathsf{PAb} \to \mathsf{CAlg}$

$$\mathcal{A} := \mathfrak{CCR} \circ \mathcal{O} : \mathsf{Man}_2 \to \mathsf{CAlg}$$

- We construct a $*$-algebra $\Delta :=$span $\{W_{[\varphi]} \,|\, [\varphi] \in H^1_c(\Sigma;\mathbb{Z})\}$ where $W_{[\,\cdot\,]}$ satisfy

$$W_{[\varphi]}\, W_{[\widetilde{\varphi}]} := e^{-i\hbar\,\tau_\Sigma([\varphi],[\widetilde{\varphi}])}\; W_{[\varphi]+[\widetilde{\varphi}]} \quad , \qquad {W_{[\varphi]}}^* := W_{-[\varphi]} \quad .$$

- We obtain a C^*-algebra taking the completion of Δ with respect to the norm

$$||a||^{m.r.} := \sup_{\omega\in\mathcal{F}} \sqrt{\omega(a^* a)}$$

where $\omega : \Delta \to \mathbb{C}$ is a state, namely $\omega(1_\Delta) = 1$ and $\omega(a^* a) \geq 0$.

- We assume that $\hbar \notin 2\pi\mathbb{Z}$ to avoid commutative C^*-algebras
- The C^*-algebra homomorphism $\mathcal{A}(f) : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ is specified by

$$\mathcal{A}(f)\big(W_{[\,\cdot\,]}\big) := W'_{f_*[\,\cdot\,]} \quad ,$$

where by W' we denote the Weyl symbols in $\mathcal{A}(\Sigma')$.

Invariant functionals on compact surfaces

- Any object Σ in Man_2 comes together with its automorphism $\mathrm{Diff}^+(\Sigma)$ of Σ
- Because $H^1_c(\Sigma;\mathbb{Z})$ is discrete, $\mathrm{Diff}^+_0(\Sigma) \subseteq \mathrm{Diff}^+(\Sigma)$ is represented trivially

$$\mathrm{MCG}(\Sigma) := \frac{\mathrm{Diff}^+(\Sigma)}{\mathrm{Diff}^+_0(\Sigma)} \to \mathrm{Aut}\big(\mathcal{A}(\Sigma)\big) \qquad f \mapsto \mathcal{A}(f)$$

- For compact Σ, there exists a short exact sequence of groups

$$1 \longrightarrow \mathrm{Tor}(\Sigma) \longrightarrow \mathrm{MCG}(\Sigma) \longrightarrow \mathrm{Sp}(H^1(\Sigma;\mathbb{Z}),\tau_\Sigma) \longrightarrow 1$$

- The representation of the $\mathrm{MCG}(\Sigma)$ descends to a representation of $\mathrm{Sp}(H^1(\Sigma;\mathbb{Z}),\tau_\Sigma)$

$$\mathrm{Sp}(H^1(\Sigma;\mathbb{Z}),\tau_\Sigma) \longrightarrow \mathrm{Aut}(\mathcal{A}(\Sigma)) \ , \quad T \longmapsto \kappa_T : W_{[\varphi]} \mapsto W_{T[\varphi]}$$

- An invariant functional under the action of the symplectic group

$$\omega\big(W_{T[\varphi]}\big) = \omega\big(W_{[\varphi]}\big) := \begin{cases} 1 & \text{if } [\varphi] = 0 \\ K_{[\varphi]} & \text{else} \end{cases}$$

- Since $\mathrm{Sp}(H^1(\Sigma;\mathbb{Z}),\tau_\Sigma) \not\subset \mathrm{U}(H^1(\Sigma;\mathbb{Z}),\mu)$, *does not* exist invariant Gaussian state

$$\omega\big(W_{[\varphi]}\big) = e^{-\mu([\varphi],[\varphi])}$$

Non-existence of natural states

No-go theorem: There exists *no natural state* or the functor $\mathcal{A} : \mathsf{Man}_2 \to \mathsf{CAlg}$, namely a state for each Σ such that for all Man_2-morphisms $f : \Sigma \to \Sigma'$ holds true:

$$\omega_{\Sigma'} \circ \mathcal{A}(f) = \omega_\Sigma$$

Sketch of the proof

- Let us assume that there exists a natural state $\{\omega_\Sigma\}_{\Sigma \in \mathsf{Man}}$
- Consider the Man_2-diagram: $\mathbb{S}^2 \xleftarrow{f_1} \mathbb{R} \times \mathbb{T} \xrightarrow{f_2} \mathbb{T}^2$
- The naturality of the state implies: $\omega_{\mathbb{S}^2} \circ \mathcal{A}(f_1) = \omega_{\mathbb{R}\times\mathbb{T}} = \omega_{\mathbb{T}^2} \circ \mathcal{A}(f_2)$
- Because of $H^1_c(\mathbb{S}^2; \mathbb{Z}) = 0$, then $\mathcal{A}(\mathbb{S}^2) \simeq \mathbb{C}$ and hence $\omega_{\mathbb{S}^2} = \mathrm{id}_\mathbb{C}$ is unique on $\mathbb{C}$
- We can choose f_2 such that $W_n^{\mathbb{R}\times\mathbb{T}} \mapsto W^{\mathbb{T}^2}_{(n,0)}$ we obtain that $\omega_{\mathbb{T}^2}\big(W^{\mathbb{T}^2}_{(n,0)}\big) = 1$
- Choosing $a = \alpha_1 1 + \alpha_2 W^{\mathbb{T}^2}_{(1,1)} + \alpha_3 W^{\mathbb{T}^2}_{(0,1)} \in \mathcal{A}(\mathbb{T}^2)$ the functional $\omega_{\mathbb{T}^2}(a^* a) < 0$

Q.E.D