The Cauchy problem for the Dirac operator on a Lorentzian spin manifold

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EINSTEIN 1915

Gravitation interaction $\longleftrightarrow$ Lorentzian manifold $(\mathcal{M}, g)$

$$\mathbf{Ric} + \mathbf{g}\,(\Lambda - \frac{1}{2}\mathbf{scal}) = \frac{8\pi G}{c^4}\mathbf{T}$$

GEOMETRY: **Ric**: Ricci (0,2)-tensor, **scal**: scalar curvature

MATTER: **T**: stress-egenergy (0,2)-tensor

PHYSICS: Λ: cosmological constant, G: gravitational constant , c: speed of light

(using the contracted) BIANCHI'S IDENTITY

$$\mathrm{div}(\mathbf{Ric} - \frac{\mathbf{scal}}{2}\,\mathbf{g}) = 0 \qquad \longrightarrow \qquad \mathrm{div}(\mathbf{T}) = 0$$

$$\mathbf{g}^{\alpha\gamma}\nabla_\gamma(\mathbf{R}_{\alpha\beta} - \tfrac{1}{2}\,\mathbf{g}_{\alpha\beta}\mathbf{R}) = \mathbf{0} \qquad \underbrace{g^{\alpha\gamma}\nabla_\gamma \mathbf{T}_{\alpha\beta} = 0}_{\text{PDEs}}$$

GOAL: Well-posedness of the Cauchy problem for the Dirac operator

Outline of the Talk

- **Mathematical Preliminaries**
  - **Lorentzian Manifolds: the Spacetime's Geometry**
  - **Spin Geometry in a Nutshell**
- **The Cauchy Problem for the Dirac Operator**
  - **Existence and Uniqueness in a Time Strip**
  - **Global Well-Posedness**
- **Outlook**

- Based on :
  The well-posedness of the Cauchy problem for the Dirac operator on globally hyperbolic manifolds with timelike boundary, Nadine Große and S.M. (arXiv:1806.06544 [math.DG])

Lorentzian Manifolds: the Spacetime's Geometry

Given a Lorentzian manifold $(\mathcal{M}, g)$ we denote

- $v \in T_p\mathcal{M}$: *spacelike* if $g(v,v) > 0$, *lightlike* if $g(v,v) = 0$, *timelike* if $g(v,v) < 0$
- $\gamma : I \to \mathcal{M}$: *spacelike* if $g(\dot\gamma,\dot\gamma) > 0$, *lightlike* if $g(\dot\gamma,\dot\gamma) = 0$, *timelike* if $g(\dot\gamma,\dot\gamma) < 0$
- *future/past* $J^\pm(p) = \{p\} \cup \{q \in \mathcal{M} : \text{future/past directed causal curve from } p \text{ to } q\}$

Definition: Let $\mathcal{M}$ of a connected, time-oriented, oriented Lorentzian manifold

- **Cauchy hypersurface** Σ: if each inextendible timelike curve $\gamma \cap \Sigma = \{\text{pt}\}$
- **Globally hyperbolic**: $\mathcal{M}$ *strongly causal* and $\forall p, q \in \mathcal{M}, J^+(p) \cap J^-(q)$ compact

Bernal-Sánchez's Theorem: Then the following are equivalent.

(i) $\mathcal{M}$ is globally hyperbolic;

(ii) There exists a Cauchy hypersurface $\Sigma \subset \mathcal{M}$;

(iii) $\mathcal{M}$ isometric to $\mathbb{R} \times \Sigma$ with metric $-\beta^2 dt^2 + h_t$, where $\beta \in C^\infty(\mathcal{M}, (0,\infty)$
  - h_t is a Riemannian metric on Σ depending smoothly on $t \in \mathbb{R}$
  - all sets $\{t_0\} \times \Sigma$ are Cauchy hypersurfaces in $\mathcal{M}$

Example: Minkoski spacetime $(\mathbb{R}^4, \eta)$, Schwarzchild spacetime $(\mathbb{R}^2 \times \mathbb{S}^2, g_S)$

NOT Example: anti-de Sitter space $(\mathbb{S}^1 \times \mathbb{R}^3, g_{adS})$, Gödel universe $(\mathbb{R}^4, g_G)$

Spin Geometry in a Nutshell

Definition: $\mathcal{M}$ be a connected, time-oriented, oriented, $n+1$-dim Lorentzian manifold

- **Spinor bundle** $S\mathcal{M}$: complex vector bundle with $N := 2^{\lfloor \frac{n+1}{2} \rfloor}$-dimensional fibers endowed with **fiberwise** pairing given by the canonical scalar product on $\mathbb{C}^N$

$$\langle \cdot \,|\, \cdot \rangle \colon S_p\mathcal{M} \times S_p\mathcal{M} \to \mathbb{C}$$

  and a **clifford multiplication**: fiber-preserving map $\gamma\colon T\mathcal{M} \to \mathrm{End}(S\mathcal{M})$

- **Spin Manifold**: manifold which admits a spinor bundle
- **Dirac operator**: $\mathsf{D}\colon \Gamma(S\mathcal{M}) \to \Gamma(S\mathcal{M})$ which in local coordinates this reads as

$$\mathsf{D} = \sum_{\mu=0}^{n} \imath\gamma(e_\mu)\nabla_{e_\mu}$$

  where $(e_\mu)_{\mu=0,\dots,n}$ is a local orthonormal Lorentzian frame of $T\mathcal{M}$ and $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u,v)$ for every $u, v \in T_p\mathcal{M}$ and $p \in \mathcal{M}$.

Remarks:

(i) Topological obstruction to existence of a spinor bundle;

(ii) Existence of spinor bundles on parallelizable manifolds;

(iii) The Dirac Cauchy problem is well posed on glob. hyp. spin manifolds with $\partial\mathcal{M} = \emptyset$

Our Setting: Globally Hyperbolic Spin Manifolds with Nonempty Boundary

- Let $(\widetilde{\mathcal{M}}, g)$ be a globally hyperbolic spin manifold of dimension $n+1 \geq 3$
- Let $(\mathcal{N}, g|_{\mathcal{N}})$ be a submanifold of $(\widetilde{\mathcal{M}}, g)$ that is itself globally hyperbolic
- Let $\widetilde{\Sigma}$ be a smooth spacelike Cauchy surface of $\widetilde{\mathcal{M}}$
- Then, $\widehat{\Sigma} := \widetilde{\Sigma} \cap \mathcal{N}$ is a spacelike Cauchy surface for $\mathcal{N}$
- We assume that $\mathcal{N}$ divides $\widetilde{\mathcal{M}}$ into two connected components
- The closure of one of them we denote by $\mathcal{M}$

Definition: We call $\mathcal{M}$ *globally hyperbolic manifold with timelike boundary*

- On $\widetilde{\mathcal{M}}$ we choose a Cauchy time function $t\colon \widetilde{\mathcal{M}} \to \mathbb{R}$
- Then $\{t^{-1}(s)\}_{s\in\mathbb{R}}$ gives a foliation by Cauchy surfaces
- We set $\Sigma_s := t^{-1}(s) \cap \mathcal{M}$.
- For $n+1=2$, $\mathcal{M}$ is homeomorphic to $\mathbb{R}\times[a,\infty)$ or $\mathbb{R}\times[a,b]$)

MAIN THEOREM

- $(\mathcal{M}, g)$ be a *globally hyperbolic spin manifold* with *timelike boundary* $\partial\mathcal{M}$;
- $S\mathcal{M} \to \mathcal{M}$ be the *spinor bundle* and $\mathrm{D} : \Gamma(S\mathcal{M}) \to \Gamma(S\mathcal{M})$ be *Dirac operator*;
- *linear, non-invertible* $\mathrm{M} : \Gamma(S\partial\mathcal{M}) \to \Gamma(S\partial\mathcal{M})$ with *constant kernel dimension* s.t.

$$\mathrm{M}\psi|_{\partial\mathcal{M}} = 0 \;\text{ and }\; \mathrm{M}^\dagger\psi|_{\partial\mathcal{M}} = 0 \quad \Longrightarrow \quad \langle\psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\psi\rangle_q = 0\,.$$

Then the Cauchy problem for the Dirac operator is **well-posed**:

(I) $\forall f \in \Gamma_{cc}(S\mathcal{M})$ and $\forall h \in \Gamma_{cc}(S\Sigma_0)$ exists a unique $\psi \in \Gamma_{sc}(S\mathcal{M})$ such that

$$\begin{cases} \mathrm{D}\psi = f \\ \psi|_{\Sigma_0} = h \\ \mathrm{M}\psi|_{\partial\mathcal{M}} = 0 \end{cases} \tag{1}$$

(II) moreover $\Gamma_{cc}(S\mathcal{M}) \times \Gamma_{cc}(S\Sigma_0) \ni (f, h) \mapsto \psi \in \Gamma_{sc}(S\mathcal{M})$ is continuous;

Example: MIT boundary condition $\mathrm{M} = (\gamma(\mathtt{n}) - \imath))$

($\gamma(\mathtt{n})$ denotes Clifford multiplication for $\mathtt{n}$, the outward unit normal on $\partial\mathcal{M}$)

Remark: The Cauchy problem (1) is still well-posed for $(f, h) \in \Gamma_c(S\mathcal{M}) \times \Gamma_c(S\Sigma_0)$

Reformulation of the Cauchy Problem I

Symmetric Positive Hyperbolic Systems

- $E \to \mathcal{M}$ be a complex vector bundle with finite rank N and fiberwise metric $\langle \cdot \,|\, \cdot \rangle$
- $\mathfrak{L} : \Gamma(E) \to \Gamma(E)$ with formal L^2-adjoint $\mathfrak{L}^\dagger$

$$(\cdot \,|\, \cdot)_{\mathcal{M}} := \int_{\mathcal{M}} \langle \cdot \,|\, \cdot \rangle \mathrm{Vol}_{\mathcal{M}} \,,$$

Definition: a 1^{st} order $\mathfrak{L}$ is called **symmetric positive hyperbolic system** if

(S) $\sigma_{\mathfrak{L}}(\xi) \colon E_p \to E_p$ is Hermitian with respect to $\langle \cdot \,|\, \cdot \rangle$, $\forall \xi \in T_p^*\mathcal{M}$ and $\forall p \in \mathcal{M}$.

(P) $\langle (\mathfrak{L} + \mathfrak{L}^\dagger) \cdot \,|\, \cdot \rangle$ on E_p is positive definite

(H) $\langle \sigma_{\mathfrak{L}}(\tau) \cdot \,|\, \cdot \rangle$ is positive definite on E_p, for any future-directed timelike $\tau \in T_p^*\mathcal{M}$

In local coordinates $(t, x^1, \dots, x^n)$ on $\mathcal{M}$ and a local trivialization of E:

$$\mathfrak{L} := A_0(p)\partial_t + \sum_{j=i}^{n} A_j(p)\partial_{x^j} + B(p) \qquad A_0, A_j, B \in C^\infty(\mathcal{M}, Mat(N \times N))$$

$$(S)\; A_0 = A_0^\dagger \,,\; A_j = A_j^\dagger \quad (P)\; \kappa := \mathfrak{L} + \mathfrak{L}^\dagger = B - \partial_t(\sqrt{g})A_0) - \sum_{j=1}^{n} \partial_{x^j}(\sqrt{g}A_j) > 0$$

$$(H)\; \sigma_{\mathfrak{L}}(\tau) = A_0 + \sum_{j=1}^{N-1} \alpha_j A_j > 0 \qquad \text{for any } \tau = dt + \sum_j \alpha_j dx^j$$

Reformulation of the Cauchy Problem II

NOT Example: $\mathscr{M}^4 := \mathbb{R}^3 \times [0,\infty)$ endowed with the element line

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2\,.$$

For the Dirac operator $\mathsf{D} = \imath\gamma(e_0)\partial_t + \imath\gamma(e_1)\partial_x + \imath\gamma(e_2)\partial_y + \imath\gamma(e_3)\partial_z$ we have

(S) $\gamma(e_j)^\dagger = -\gamma(e_j)$ ↯ (P) $\kappa = 0$ ↯ (H) $\sigma_{\mathsf{D}}(dt) = \gamma(e_0) \not> 0$ ↯

Lemma 1: Let be $\mathfrak{S} : \Gamma(S\mathcal{M}) \to \Gamma(S\mathcal{M})$ defined by $\mathfrak{S} = -\imath\gamma(e_0)\mathsf{D} + \lambda\,\mathrm{Id}$. Then:

(I) $\mathfrak{S}$ is symmetric hyperbolic system for all $\lambda \in \mathbb{R}$

(II) Its Cauchy problem is equivalent to the Cauchy problem for the Dirac operator

$$\begin{cases} \mathsf{D}\psi = f \in \Gamma_c(S\mathcal{M}) \\ \psi|_{\Sigma_0} = h \in \Gamma_c(S\Sigma_0) \\ \mathsf{M}\psi|_{\partial\mathcal{M}} = 0. \end{cases} \iff \begin{cases} \mathfrak{S}\Psi = \mathfrak{f} \in \Gamma_c(S\mathcal{M}) \\ \Psi|_{\Sigma_0} = \mathfrak{h} \in \Gamma_c(S\Sigma_0) \\ \mathsf{M}\Psi|_{\partial\mathcal{M}} = 0 \end{cases} \tag{2}$$

(III) $\forall\ \mathcal{R} \subset \mathcal{M}$ compact $\exists\ \lambda > 0$ s. t. $\mathfrak{S}$ is a symmetric positive hyperbolic system.

Idea of Proof of (II): $\Psi = e^{-\lambda t}\psi \implies \mathfrak{h} = e^{-\lambda t}h$, $\mathfrak{f} = e^{-\lambda t}\gamma(e_0)f$ and

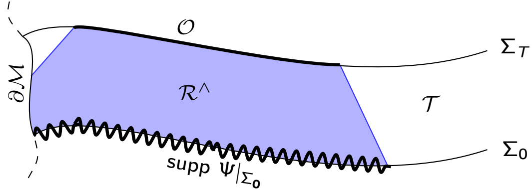
$$\mathfrak{S}\Psi = \mathfrak{S}(e^{-\lambda t}\psi) = (-\imath\gamma(e_0)\mathsf{D} + \lambda\mathrm{Id})(e^{-\lambda t}\psi) = -\imath e^{-\lambda t}\gamma(e_0)\mathsf{D}\psi = e^{-\lambda t}\gamma(e_0)f.$$

$$\mathsf{M}\Psi|_{\partial\mathcal{M}} = e^{-\lambda t}\mathsf{M}\psi|_{\partial\mathcal{M}} = 0 \quad \text{if and only if} \quad \mathsf{M}\psi|_{\partial\mathcal{M}} = 0.$$

Energy Inequality in a Time Strip

- Time strip: $\mathcal{T} := t^{-1}([0, T])$ where $t\colon \mathcal{M} \to \mathbb{R}$ is the Cauchy time function
- Let $\lambda \in \mathbb{R}$ s.t. $\mathfrak{S} = -\imath\gamma(e_0)\mathsf{D} + \lambda$ is a *symmetric positive hyperbolic system* on

$$\mathcal{R}^\wedge := J^-(\mathcal{O}) \cap \mathcal{T}$$

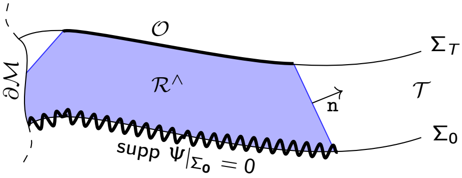


Lemma 2: Let $\Psi \in \Gamma(S\mathcal{T})$ satify $\Psi|_{\Sigma_0} = 0$ and $\mathsf{M}\Psi|_{\partial\mathcal{M}} = 0$. Then Ψ satisfies the

Energy Inequality $\|\Psi\|_{L^2(\mathcal{R}^\wedge)} \leq c\|\mathfrak{S}\Psi\|_{L^2(\mathcal{R}^\wedge)}$ for constant $c > 0$ independent on Ψ.

Sketch of the proof of Lemma 2

(Now we use that $\mathfrak{S}$ is a Symmetric Positive Hyperbolic system)



-**(S)** $\Rightarrow$ Green identity: $\quad (\Psi \,|\, \mathfrak{S}\Psi)_{\mathcal{R}^\wedge} - (\mathfrak{S}^\dagger\Psi \,|\, \Psi)_{\mathcal{R}^\wedge} = (\Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi)_{\partial\mathcal{R}^\wedge}$

$$\underbrace{(\Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi)_{\partial\mathcal{R}^\wedge}}_{\text{we want to estimate}} - 2(\Psi \,|\, \mathfrak{S}\Psi)_{\mathcal{R}^\wedge} = -(\Psi \,|\, \mathfrak{S}\Psi)_{\mathcal{R}^\wedge} - (\Psi \,|\, \mathfrak{S}^\dagger\Psi)_{\mathcal{R}^\wedge}$$

$$= -(\Psi \,|\, (\mathfrak{S} + \mathfrak{S}^\dagger)\Psi)_{\mathcal{R}^\wedge} \overset{\textbf{(P)}}{\leq} -2c(\Psi \,|\, \Psi)_{\mathcal{R}^\wedge}$$

- Boundary: $\partial\mathcal{R}^\wedge = \mathcal{O} \cup \Big(\Sigma_0 \cap J^-(\mathcal{O})\Big) \cup Y$, and $Y = (Y \cap \partial\mathcal{M}) \sqcup (Y \setminus (Y \cap \partial\mathcal{M}))$

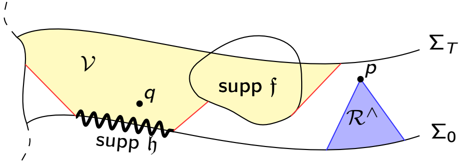
- **(H)**$\Rightarrow$ $(\Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi)_{\mathcal{O}} > 0$ and $(\Psi \,|\, \gamma(e_0)\gamma(\mathtt{n})\Psi)_{\mathcal{Y}\setminus(\mathcal{Y}\cap\partial\mathcal{M})} \geq 0$

- Hence: $\quad 2(\Psi \,|\, \lambda\Psi)_{\mathcal{R}^\wedge} \leq 2(\Psi \,|\, \mathfrak{S}\Psi)_{\mathcal{R}^\wedge} \xRightarrow{\text{Hölder ineq.}} \|\Psi\|_{L^2(\mathcal{R}^\wedge)} \leq \lambda^{-1}\|\mathfrak{S}\Psi\|_{L^2(\mathcal{R}^\wedge)}$

Finite Propagation of Speed

Proposition 3: Any solution ψ to the Dirac Cauchy problem (1) propagates with at most speed of light, i.e. its support on $\mathcal{T}$ is inside the region

$$\mathcal{V} := \Big(J^+(\text{supp } f \cap \mathcal{T}) \cup J^+(\text{supp } h)\Big) \cap \mathcal{T},$$

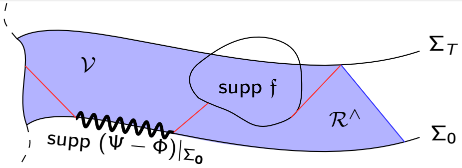


Proof:

- Choose λ s.t. $\mathfrak{S}$ is a symmetric positive hyperbolic system on $\mathcal{R}^\wedge = \mathcal{T} \cap J^-(p)$
- $\mathfrak{h}|_{\mathcal{R}^\wedge \cap \Sigma_0} \equiv 0$ and Lemma 2 $\Rightarrow \|\Psi\|_{L^2(\mathcal{R}^\wedge)} \leq c\|\mathfrak{S}\Psi\|_{L^2(\mathcal{R}^\wedge)} = 0$ in $\mathcal{R}^\wedge$
- Hence, $\Psi = 0$ outside $\mathcal{V}$.
- Lemma 1 $\Rightarrow \psi$ propagates with at most speed of light □

Uniqueness of the Solution

Proposition 4: Suppose there exist $\psi, \phi \in \Gamma(S\mathcal{T})$ satisfying the same Cauchy problem (1). Then $\psi = \phi$.



Proof:

- Lemma 1 $\Rightarrow \Psi, \Phi$ are solutions for the same Cauchy problem (2).

$$\begin{cases} \mathfrak{S}(\Psi - \Phi) = 0 \\ (\Psi - \Phi)|_{\Sigma_0} = 0 \\ M(\Psi - \Phi)|_{\partial\mathcal{M}} = 0 \end{cases}$$

- Finite Prop. Speed $\Rightarrow$ supp Ψ and supp Φ are contained in $\mathcal{R}^\wedge$ for $\mathcal{O} := \mathcal{V} \cap \Sigma_T$.
- Energy Inequality $\Rightarrow \|\Psi - \Phi\|_{L^2(\mathcal{R}^\wedge)} \leq c\|\mathfrak{S}\Psi\|_{L^2(\mathcal{R}^\wedge)} = 0$
- Hence $\Psi = \Phi \xRightarrow{\text{Lemma 1}} \psi = \phi$. □

Weak and Strong Solutions

Definition: We call $\Psi \in \mathscr{H} := \overline{(\Gamma_c(S\mathcal{T}), (.\,|\,.)_{\mathcal{T}})}^{(.\,|\,.)_{\mathcal{T}}}$

(W) **Weak Solution** if it holds $(\Phi\,|\,\mathfrak{f})_{\mathcal{T}} = (\mathfrak{S}^{\dagger}\Phi\,|\,\Psi)_{\mathcal{T}}$
for any $\Phi \in \Gamma_c(S\mathcal{T})$ such that $\mathrm{M}^{\dagger}\Phi|_{\partial\mathcal{M}} = 0$ and $\Phi|_{\Sigma_T} \equiv 0$

(S) **Strong Solution** if $\exists\,\{\Psi_k\}_k \subset C^{\infty}(\Gamma(SU))$ s.t. $\mathrm{M}\Psi_k = 0$ on $\partial\mathcal{M} \cap U$ and

$$\|\Psi_k - \Psi\|_{L^2(U)} \xrightarrow{k\to\infty} 0 \quad \text{and} \quad \|\mathfrak{S}\Psi_k - \mathfrak{f}\|_{L^2(U)} \xrightarrow{k\to\infty} 0$$

where $U \subset \mathcal{M}$ be a compact subset in $\mathcal{M}$.

Lemma 5: A weak solution Ψ of the Cauchy problem (2) is locally a strong solution.

Comments on the Proof of Lemma 5:

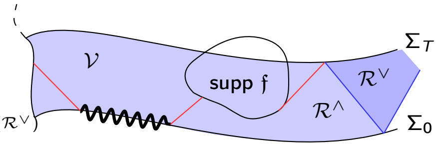
- Far from the boundary, we can use a family of mollifier to conclude
- At the boundary, we choose Fermi coordinates $(x^0, x^1, \dots, x^{n-1}, \widetilde{z})$ such that

$$\widehat{\mathfrak{S}} := (\gamma(e_0)\gamma(e_n))^{-1}\mathfrak{S} = \partial_{\widetilde{z}} + \sum_{j=0}^{n-1} A_j(x)\partial_{x^j} + B(x)$$

- Family of mollifier in $(x^0, \dots, x^{n-1})$-direction + Sobolev theory to conclude.

Existence of a Weak Solution

Theorem 6: There exists a unique weak solution $\Psi \in \mathscr{H}$ to the Cauchy problem (2) with $\mathfrak{f} \in \Gamma_{cc}(S\mathcal{M})$ and $\mathfrak{h} \equiv 0$, restricted to $\mathcal{T}$.



Sketch of the Proof:

- Fin. Prog. Speed: $(\Phi \,|\, \mathfrak{f})_{\mathcal{R}^\vee} = (\mathfrak{S}^\dagger \Phi \,|\, \Psi)_{\mathcal{R}^\vee}$
- Energy Estimates: $\|\Phi\|_{L^2(\mathcal{R}^\vee)} \leq c \|\mathfrak{S}^\dagger \Phi\|_{L^2(\mathcal{R}^\vee)}$
- The kernel of the operator $\mathfrak{S}^\dagger$ acting on $\operatorname{dom}\mathfrak{S}^\dagger$ is trivial

$$\operatorname{dom}\mathfrak{S}^\dagger := \{\Phi \in \Gamma_c(S\mathcal{T}) \mid \Phi|_{\Sigma_T} = 0, \mathrm{M}^\dagger \Phi|_{\partial\mathcal{M}} = 0\}$$

- $\ell \colon \mathfrak{S}^\dagger(\operatorname{dom}\mathfrak{S}^\dagger) \to \mathbb{C}$ given by $\ell(\Theta) = (\Phi \,|\, \mathfrak{f})_{\mathcal{R}^\vee}$ where Φ satisfies $\mathfrak{S}^\dagger \Phi = \Theta$
- Energy Estimates $\Rightarrow \ell$ is bounded:

$$\ell(\Theta) = (\Phi \,|\, \mathfrak{f})_{\mathcal{R}^\vee} \leq \|\mathfrak{f}\|_{L^2(\mathcal{R}^\vee)} \|\Phi\|_{L^2(\mathcal{R}^\vee)} \qquad \text{(Cauchy-Schwarz inequality)}$$

$$\leq \lambda^{-1} \|\mathfrak{f}\|_{L^2(\mathcal{R}^\vee)} \|\mathfrak{S}^\dagger \Phi\|_{L^2(\mathcal{R}^\vee)} = \lambda^{-1} \|\mathfrak{f}\|_{L^2(\mathcal{R}^\vee)} \|\Theta\|_{L^2(\mathcal{R}^\vee)},$$

-Hence $(\Phi \,|\, \mathfrak{f})_{\mathcal{R}^\vee} = \ell(\Theta) \overset{Riesz}{\underset{Thm.}{=}} (\Theta \,|\, \Psi)_{\mathcal{R}^\vee} = (\mathfrak{S}^\dagger \Phi \,|\, \Psi)_{\mathcal{R}^\vee}$ for all $\Phi \in \operatorname{dom}\mathfrak{S}^\dagger$

Global Existence and Green Operators

Sketch of part (I) of the MAIN THEOREM:

- for any $T \in [0, \infty)$ exists a unique $\psi_T \in \Gamma(S\mathcal{T}_T)$ of the Dirac Cauchy problem (1)
- For any $T_1, T_2 \in [0, \infty)$ with $T_2 > T_1 \xrightarrow[sol.]{unique} \psi_{T_2}|_{\mathcal{T}_{T_1}} = \psi_{T_1}$.
- Hence, we can glue everything together to obtain a smooth solution for all $\mathcal{T} \geq 0$
- A similar arguments holds for negative time.
- Since $h \in \Gamma_{cc}(S\Sigma_0)$, $f \in \Gamma_{cc}(S\mathcal{M}) \xrightarrow[Speed]{Fin.\ Prop.}$ the solution is spacelike compact. □

Proposition 7: The Dirac operator is Green hyperbolic. i.e. there exist linear maps *advanced/retarded Green operator* $\mathrm{G}^{\pm} : \Gamma_{cc}(S\mathcal{M}) \to \Gamma_{sc}(S\mathcal{M})$ satisfying

(i) $\mathrm{G}^{\pm} \circ \mathrm{D} f = \mathrm{D} \circ \mathrm{G}^{\pm} f = f$ for all $f \in \Gamma_{cc}(S\mathcal{M})$;

(ii) $\mathrm{supp}\,(\mathrm{G}^{\pm} f) \subset J^{\pm}(\mathrm{supp}\, f)$ for all $f \in \Gamma_{cc}(S\mathcal{M})$,

where $J^{\pm}$ denote the causal future (+) and past (-).

Outlook

WHAT WE HAVE SEEN AND WHAT COMES NEXT?

– well-posedness of the Cauchy problem for

✓ Dirac equation with local boundary condition (Nadine Große)

? Dirac equation with nonlocal boundary condition (Nicolò Drago & Nadine Große)

? Wave equation (with Nicolas Ginoux & Nadine Große)

? Maxwell equation (with Nicolas Ginoux & Nadine Große)

ADDITIONAL DIFFICULTIES:

– reduce wave equation and maxwell equation to 1^{st}-order systems

Q: Are those systems symmetric, hyperbolic and positive?

– $\partial\mathcal{M}$ is **characteristic** for the 1^{st}-order systems: $\det \not{n} = 0$ where $n \perp \partial\mathcal{M}$

Q: weak solution=strong solution?