Fermionic Projectors and Hadamard States

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``To the memory of Rudolf Haag, one of the founding fathers of the algebraic quantum field theory''

Outline

- **The algebraic approach to QFT**
- **Quasi-free states for CAR algebras**
- **Fermionic Projector and the Hadamard condition:**
  - **FP states in a strip of spacetime**
  - **FP states and the mass oscillation property**
  - **FP states via the Møller operator**
- **Conclusions**

Based on:

- F. Finster, S. M., C. Röken: *"The Fermionic Projector in a time-dependent external potential: mass oscillation property and Hadamard states"*
- N. Drago, S. M.: *"A new class of Fermionic Projectors: Møller operators and mass oscillation properties."* (work in progress)

Algebraic Quantum Field Theory

AQFT - I: Dirac field

- **Spinor bundle** $S\mathcal{M} \simeq \mathcal{M} \times \mathbb{C}^4$ and **cospinor bundle** $S^*\mathcal{M} \simeq \mathcal{M} \times (\mathbb{C}^4)^*$, with $\mathcal{M} \simeq \mathbb{R} \times \Sigma$ 4-dim *globally hyperbolic spacetime* :

$$ds^2 = \beta^2 dt^2 - h_t; \quad \beta \in C^\infty(M; \mathbb{R}^+) \text{ and } h_t \in \mathrm{Riem}(\Sigma); \forall t \in \mathbb{R}.$$

- Spinors $\psi \in C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4)$ and cospinors $\varphi \in C^\infty_{sc}(\mathcal{M}, (\mathbb{C}^4)^*)$.

- Dirac conjugation map: $A \;:\; C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \underset{(\leftarrow)}{\longrightarrow} C^\infty_{sc}(\mathcal{M}, (\mathbb{C}^4)^*)$.

- **Spin scalar product**: $\prec \cdot \mid \cdot \succ_x : C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \times C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \to \mathbb{C}$

$$\prec \psi | \widetilde{\psi} \succ_x := ((A\psi)\widetilde{\psi})(x).$$

- **spacetime inner product**: $<\cdot|\cdot> \;:\; C^\infty_{sc}(\mathcal{M}, \mathbb{C}^4) \times C^\infty_{c}(\mathcal{M}, \mathbb{C}^4) \to \mathbb{C}$

$$<\psi|\vartheta> = \int_{\mathcal{M}} \prec \psi | \vartheta \succ_x d\mu_g \,.$$

AQFT - II: Dynamics

- **Dirac operator** on SM and its dual on S^*M:

$$\mathcal{D}\psi_m \doteq (i\gamma^\mu \nabla_\mu + B - m)\psi_m = 0, \qquad \mathcal{D}^*\varphi_m = (-i\gamma^\mu \nabla_\mu + B - m)\varphi_m = 0.$$

- **Causal propagators:** $G^{(*)} : C_c^\infty(\mathcal{M}, (\mathbb{C}^4)^{(*)}) \to C_{sc}^\infty(\mathcal{M}, (\mathbb{C}^4)^{(*)})$

$$\mathcal{D}^{(*)} \circ G^{(*)} = 0 = G^{(*)} \circ \mathcal{D}^{(*)}|_{C_c^\infty(\mathcal{M},(\mathbb{C}^4)^{(*)})}$$

$$supp(G^{(*)}(f)) \subseteq J(supp(f)), \qquad \forall f \in C_c^\infty(\mathcal{M}, (\mathbb{C}^4)^{(*)})$$

- **Hilbert spaces:**

$$\mathcal{H}_m^s := \overline{\left(\mathcal{S}ol(\mathcal{D}) \simeq \frac{C_c^\infty(\mathcal{M}, \mathbb{C}^4)}{\mathcal{D}C_c^\infty(\mathcal{M}, \mathbb{C}^4)}, (\cdot \mid \cdot)_m^s \doteq \int_\Sigma \prec \cdot \,|\not\psi\, \cdot \succ_x d\Sigma\right)}$$

$$\mathcal{H}_m^c := \overline{\left(\mathcal{S}ol(\mathcal{D}^*)) \simeq \frac{C_c^\infty(\mathcal{M}, (\mathbb{C}^4)^*)}{\mathcal{D}^* C_c^\infty(\mathcal{M}, \mathbb{C}^4)^*)}, (\cdot \mid \cdot)_m^c \doteq \int_\Sigma \prec A^{-1} \cdot \,|\not\psi\, A^{-1} \cdot \succ_x d\Sigma\right)}$$

AQFT - III: CAR Algebra

- unital Borchers-Uhlmann $*$-algebra: $\mathcal{A} = \bigoplus_{k=0}^{\infty} \left(\mathcal{S}ol(\mathcal{D}_m) \oplus \mathcal{S}ol(\mathcal{D}_m^*) \right)^{\otimes k}$
  - $\mathbb{1} = \{1, 0, 0, \ldots\}, \Phi(\psi_m) = \left\{0, \begin{pmatrix} \psi_m \\ 0 \end{pmatrix}, 0, \ldots \right\}, \Psi(\varphi_m) = \left\{0, \begin{pmatrix} 0 \\ \varphi_m \end{pmatrix}, 0, \ldots \right\}$
  - $*$-operation: $\left\{0, 0, \begin{pmatrix} \psi_m \\ \varphi_m \end{pmatrix} \otimes \begin{pmatrix} \widetilde{\psi}_m \\ \widetilde{\varphi}_m \end{pmatrix}, \ldots \right\}^* = \left\{0, 0, \begin{pmatrix} A^{-1}\widetilde{\varphi}_m \\ A\widetilde{\psi}_m \end{pmatrix} \otimes \begin{pmatrix} A^{-1}\varphi_m \\ A\psi_m \end{pmatrix}, \ldots \right\}$
- We encode the CARs in the $*$-ideal $\mathcal{I} \subset \mathcal{A}$ generated by:
  - $\Phi(\psi_m) \otimes \Phi(\widetilde{\psi}_m) + \Phi(\widetilde{\psi}_m) \otimes \Phi(\psi_m)$
  - $\Psi(\varphi_m) \otimes \Psi(\widetilde{\varphi}_m) + \Psi(\widetilde{\varphi}_m) \otimes \Psi(\varphi_m)$
  - $\Psi(\varphi_m) \otimes \Phi(\psi_m) + \Phi(\psi_m) \otimes \Psi(\varphi_m) - \left(A^{-1}\varphi_m \mid \psi_m\right)_m^s \mathbb{1}$
- **Algebra of fields:** $\mathcal{F} \doteq \dfrac{\mathcal{A}}{\mathcal{I}}$

AQFT - IV: States

- **Algebraic state** $\omega : \mathcal{F} \to \mathbb{C}$ such that: $\omega(\mathbb{1}) = 1, \qquad \omega(h^* h) \geq 0, \quad \forall\, h \in \mathcal{F}$.

N.B.: Choosing a state ω is equivalent to assigning $\omega_n(h_1, \ldots, h_n)$ $\forall n \in \mathbb{N}$ and $\forall h_i \in \mathcal{F}$.

- **Quasi-free states:** $\qquad \omega_{2n+1}(h_1, \ldots, h_{2n+1}) = 0$

$$\omega_{2n}(h_1, \ldots, h_{2n}) = \sum_{\sigma \in S'_{2n}} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^{n} \omega_2\big(h_{\sigma(2i-1)}, h_{\sigma(2i)}\big).$$

Question: *Are all states physically acceptable?*

Of course not! Minimal physical requirements are:

i) covariant construction of Wick polynomials to deal with interactions,

ii) same UV behaviour of the Minkowski vacuum,

iii) finite quantum fluctuations of all observables.

Answer: **Hadamard States**

AQFT - V: Hadamard States

- A (quasi-free) state ω satisfies the **Hadamard condition** if and only if

$$WF(\omega_2) = \left\{(x, y, \xi_x, \xi_y) \in T^*M^{\otimes 2} \setminus 0 \,\middle|\, (x, \xi_x) \sim (y, -\xi_y), \quad \xi_x \triangleright 0\right\}.$$

Question: *How many Hadamard states do we know?*

- deformation arguments
- static spacetime
- pseudodifferential calculus
- holographic techniques

Question: *Does there exist an explicit method that which does not make use of use symmetries?*

Quasi-free states for CAR algebras

Araki's characterisation

[H. Araki: *"On quasifree states of CAR and Bogoliubov automorphisms"*.]

Lemma 3.2: For any quasi-free state ω on $\mathcal{F}$ there exists $Q \in \mathcal{B}(\mathscr{H}_m^s)$ satisfying

(a) $\omega_2\Big(\Psi(\varphi_m)\Phi(\psi_m)\Big) = \big(A^{-1}\varphi_m \mid Q\psi_m\big)_m^s$

(b) $Q + AQA = \mathbb{1}$,

(c) $0 \leq Q = Q^* \leq \mathbb{1}$.

Lemma 3.3: Let Q be a bounded symmetric operator on $\mathscr{H}_m^s$ with the following properties

(a) $Q + AQA = \mathbb{1}$,

(b) $0 \leq Q = Q^* \leq \mathbb{1}$.

Then there exists a unique quasi-free state ω on $\mathcal{F}$ such that

$$\omega_2\Big(\Psi(\varphi_m)\Phi(\psi_m)\Big) = \big(A^{-1}\varphi_m \mid Q\psi_m\big)_m^s$$

- Our motto will be $\Longrightarrow$ "Split the $\mathscr{H}$ilbert space!" (...but how?)

The Fermionic Signature Operator

- As before $\mathcal{H}^s_m := \overline{\left(\mathcal{S}ol(\mathcal{D})\,,\, (\cdot \mid \cdot)^s_m\right)}$ and let $N(\cdot\,,\cdot) : \mathcal{H}^s_m \times \mathcal{H}^s_m \to \mathbb{C}$ be a symmetric and densely defined sesquilinear form satisfying :

  a) $N(\cdot,\cdot) \leq C(\cdot)||\cdot||.$

- The **Fermionic Signature Operator** is $\mathcal{S} \in \mathcal{L}(\mathcal{H}^s_m)$ built out of the Riesz theorem

$$N(\cdot\,,\cdot) = (\cdot\,|\,\mathcal{S}\,\cdot)^s_m.$$

- Taking $\mathcal{S}^2 := \mathcal{S}^*\mathcal{S}$, we construct the **Fermionic Projector**

$$\chi^{\pm}(S) := \frac{1}{2|S|}(S \pm |S|) : \mathcal{H}^s_m \to \mathcal{H}^s_m.$$

N.B.: If $N(\cdot\,,\cdot) \leq ||\cdot||\;||\cdot||$, then $\mathcal{S} \in \mathcal{B}(\mathcal{H}^s_m)$ is essentially self-adjoint and

$$\chi^{\pm}(S) = \int_{\sigma(S)} \chi(\lambda) dE_\lambda$$

Fermionic Projectors and the Hadamard condition

FP states in a strip of spacetime

- Consider $\Omega \subset (-T, T) \times \Sigma \subset \mathcal{M}$ and $\mathcal{H}^s_m(\Omega) := \overline{(\mathcal{S}ol(\mathcal{D}) \,,\, (\cdot \mid \cdot)^s_m)}$.
- The **spacetime inner product:** $<\cdot|\cdot> \;:\; \mathcal{H}^s_m(\Omega) \times \mathcal{H}^s_m(\Omega) \to \mathbb{C}$

$$<\psi_m|\varphi_m> = \int_\Omega \prec\psi_m|\varphi_m\succ_x \, d\mu_\Omega \qquad \textit{(well defined)}$$
$$|<\varphi_m|\psi_m>| \leq c \,\|\varphi_m\|_m \,\|\psi_m\|_m \qquad \textit{(bounded)}.$$

- Then the **Fermionic Signature operator** is essentially self-adjoint and

$$\chi^\pm(\mathcal{S}) = \int_\sigma \chi^\pm(\lambda)\, dE_\lambda \quad \overset{Araki}{\Longrightarrow} \quad \omega_{FP} \text{ on } \mathcal{F}.$$

Problems:

- For $T \to \infty$ the spacetime inner product is not well defined,
- $\omega_2(x, y)$ is in general not Hadamard!
  [C. Fewster and B. Lang: *"Pure quasifree states of the Dirac field from the fermionic projector"*.]

FP states and the mass oscillation property

The strong mass oscillation property

- Hilbert space: $\mathscr{H}^\infty = \left(\Psi := (\psi_m)_{m \in \mathrm{I}=(m_L, m_R)} \ , \ (\cdot\ ,\cdot) = \int_{\mathrm{I}} (\cdot|\cdot)_m \, dm \right)$
- Smearing operator: $\mathfrak{p} \ : \ \mathscr{H}^\infty \to C^\infty_{\mathrm{sc}}(\mathcal{M}, S\mathcal{M}) \ , \quad \mathfrak{p}\Psi = \int_I \psi_m \, dm$
- Symmetric sesquilinear form: $<\!\mathfrak{p} \cdot |\mathfrak{p} \cdot\!> \ : \ \mathscr{H}^\infty \times \mathscr{H}^\infty \to \mathbb{C}$

Q: *Is the new sesquilinear form bounded?*

- The **strong mass oscillation property**:

$$\left|<\!\mathfrak{p}\Psi \,|\, \mathfrak{p}\Phi\!>\right| \leq c ||\psi_m|| \, ||\varphi_m||$$
$$\left|<\!\mathfrak{p} T\Psi \,|\, \mathfrak{p}\Phi\!>\right| = \left|<\!\mathfrak{p}\Psi \,|\, \mathfrak{p} T\Phi\!>\right|$$

where $T\Psi = (m\psi_m)_{m\in I}$ and $T\Phi = (m\varphi_m)_{m\in I}$.

- Family of linear operators $(\mathcal{S}_m)_{m\in \mathrm{I}}$ with $\mathcal{S}_m \in \mathcal{B}(\mathscr{H}^s_m)$ and $\sup_{m\in I} \|\mathcal{S}_m\| < \infty$

$$m \mapsto (\psi_m \,|\, \mathcal{S}_m \, \varphi_m)_m \text{ is continuous}$$
$$<\!\mathfrak{p}\Psi|\mathfrak{p}\Phi\!> = \int_I (\psi_m \,|\, \mathcal{S}_m \, \varphi_m)_m \, dm$$

FP states and the strong mass oscillation property

[F. Finster, S. M., C. Röken: *"The Fermionic Projector in a time-dependent external potential: mass oscillation property and Hadamard states"*.]

$$i\partial_t \psi_m = -\gamma^0 (i\vec{\gamma}\vec{\nabla} + B(t,\vec{x}) - m)\psi_m =: H\psi_m \qquad \text{(Dirac equation)}$$

$$\psi_m|_t = U_m^{t,t_0}\psi_0 + i\int_{t_0}^{t} U_m^{t,\tau}\left(\gamma^0 B\,\psi_m\right)\Big|_\tau \,d\tau \qquad \text{(Lippmann-Schwinger equation)}$$

- Setting $B(t,\vec{x}) = 0$: the *strong mass oscillation property holds* and

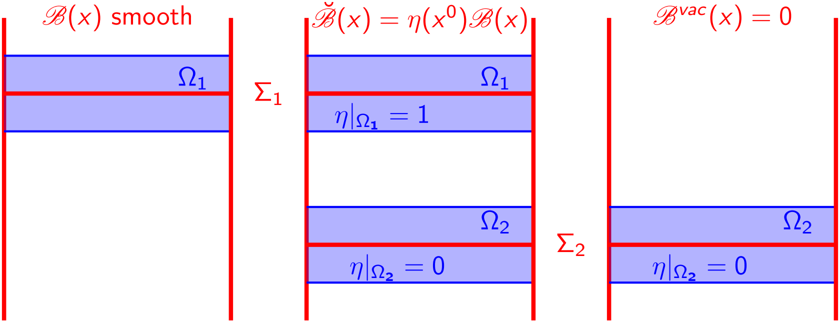
$$\mathcal{B}(\mathscr{H}_m^s) \ni \mathcal{S}_m(\vec{\xi}) := \sum_{\xi^0 = \pm\omega(\vec{\xi})} \frac{\not{\xi} + m}{2\,\xi^0(\vec{\xi})}\,\gamma^0 \Longrightarrow \underbrace{\chi^+(\mathcal{S}_m) = \Theta(\xi^0)}_{\text{frequencies splitting}}$$

- Assuming $|B(t)|_{C^2} \le c\left(1 + |t|^{2+\varepsilon}\right)^{-1}$ and $\int_{-\infty}^{\infty} |B(t)|_{C^0}\,dt < \sqrt{2} - 1$,

$$\mathcal{B}(\tilde{\mathscr{H}}_m^s) \ni \tilde{\mathcal{S}}_m \Rightarrow \chi^\pm(\tilde{\mathcal{S}}_m) = \chi^\pm(\mathcal{S}) + \underbrace{\frac{1}{2\pi i}\oint_{\partial B_{\frac{1}{2}}(\pm 1)} (\tilde{\mathcal{S}}_m - \lambda)^{-1}\,\Delta\tilde{\mathcal{S}}\,(\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1}\,d\lambda}_{\text{integral operator with smooth kernel}}\,.$$

FP states and the mass oscillation property

F. Finster, S. M., C. Röken: *"The Fermionic Projector in a time-dependent external potential: mass oscillation property and Hadamard states"*



$P = \chi^+(\mathcal{S})\, \widetilde{G}_m + (\text{smooth})$ $\quad \breve{P} = \chi^+(\mathcal{S})\, \breve{G}_m + (\text{smooth})$ $\quad P^{\mathrm{vac}} = \chi^+(\mathcal{S})\, G_m$

- $\omega_2^{\mathrm{vac}}(x,y)$ is Hadamard in Ω_0 and then in $\mathcal{M}$ via the prop. of singularities.
- $\breve{\omega}_2(x,y) - \omega_2^{\mathrm{vac}}(x,y) \in C^\infty(\mathbb{R}^4)$ for all $x, y \in \Omega_0$, since $\breve{\mathcal{G}}_m(x,y) \equiv \mathcal{G}_m(x,y)$.
- $\breve{\omega}_2(x,y)$ is Hadamard in Ω_0 and then in $\mathcal{M}$ via the prop. of singularities.
- $\omega_2(x,y) - \breve{\omega}_2(x,y) \in C^\infty(\mathbb{R}^4)$ for all $x, y \in \Omega_1$, since $\widetilde{\mathcal{G}}_m(x,y) \equiv \breve{\mathcal{G}}_m(x,y)$.
- $\omega_2(x,y)$ is **Hadamard** in Ω_1 and then in $\mathcal{M}$ via the prop. of singularities.

FP states via the Møller operator (...still work in progress)

[N. Drago, S. M.: *"A new class of Fermionic Projectors: Møller operators and mass oscillation properties".]*

- Consider $P_V := \Box - V$ and $P_{V'} := \Box - V'$

$$P_{V'} = P_V + V - V' = P_V\big(Id + E_V^{\pm}(V - V')\big)$$

  if $(V - V')$ is compact or future/past compact.

- **Møller operator:** $R^{\pm}_{V',V} := \big(Id + E_V^{\pm}(V - V')\big)^{-1} : \mathcal{S}ol(P_V) \to \mathcal{S}ol(P_{V'})$.

- Fix a Cauchy surface Σ, define $\varrho^+ = 1$ on $J^+(\Sigma_\varepsilon) \setminus \Sigma_\varepsilon$ and $\varrho^+ = 0$ on $J^-(\Sigma_\varepsilon) \setminus \Sigma_\varepsilon$, $\varrho^- = 1 - \varrho^+$ and introduce $m''(m, m') := \varrho^+ m' + m\varrho^-$

$$R^+_{m'',m} := Id - G^{\pm}_{m''}(m - m'') = Id - G^{\pm}_{m''}\varrho^+(m' - m)$$
$$R^-_{m',m''} := Id - G^{\pm}_{m'}(m'' - m') = Id - G^{\pm}_{m'}\varrho^-(m - m').$$

- **Ultra Møller operator:** $R_{m',m} := R^-_{m',m''} \circ R^+_{m'',m} : \mathscr{H}_m \to \mathscr{H}_{m'}$

new families of solutions: $\psi_m \mapsto \mathfrak{R}\psi_m := (R_{\beta,m}\psi_m)_{\beta \in I}$

FP states via the Møller operator (...still work in progress)

[N. Drago, S. M.: *"A new class of Fermionic Projectors: Møller operators and mass oscillation properties".]*

- As before $\mathcal{H}_m^s := \overline{\left(\mathcal{S}ol(\mathcal{D})\,,\,(\cdot\mid\cdot)_m^s\right)}$ and let

$$N(\cdot\,,\cdot) = <\mathfrak{p}\mathfrak{R}\;\cdot|\mathfrak{p}\mathfrak{R}\;\cdot> : \mathcal{H}_m^s \times \mathcal{H}_m^s \to \mathbb{C}$$

- If the **weak** mass oscillation property holds

$$N(\cdot,\cdot) \leq C(\cdot)||\cdot||,$$

then **Fermionic Signature Operator** is $\mathcal{S} \in \mathcal{L}(\mathcal{H}_m^s)$ built out of

$$N(\cdot\,,\cdot) = (\cdot\,|\,\mathcal{S}\,\cdot)_m^s.$$

- As before, we define the **Fermionic Projector** as

$$\chi^{\pm}(S) := \frac{1}{2|S|}(S \pm |S|).$$

Conclusions

What we know:

- Araki's lemmas: $\omega : \mathcal{F} \to \mathbb{C} \quad \Longleftrightarrow \quad Q \in \mathcal{B}(\mathscr{H}_m^s)$;
- Spectral calculus: $\chi^{\pm}(\mathcal{S})\chi^{\pm}(\mathcal{S}) = \chi^{\pm}(\mathcal{S}) \quad \chi^{\pm}(\mathcal{S})\chi^{\mp}(\mathcal{S}) = 0$.

Benefit:

- No use of symmetries;
- Distinguished states.

Price to pay:

- Not always available a sesquilinear form $\Longrightarrow$ *mass oscillation property*.

Future investigations:

- Hadamard condition.