Looking at the Hadamard property: a brief overview

Simone Murro

Fakultät für Mathematik
Universität Regensburg

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Outline

- Huygens' principle
  - Elementary Solution
- Hadamard's condition
  - Wald's hypothesis
  - Kay's conjecture
  - Radzikowski's solution
  - Sahlmann and Verch's extension

Huygens' principle

"Every point on a wave front can be considered as a new source of spherical wavelets"
[*Christiaan Huygens: Traité de la lumière. 1678*]

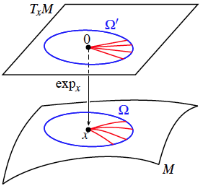
"If, at the instant $t = 0$ *-or more exactly throughout a short interval* $-\varepsilon \leqslant t \leqslant 0-$ *we produce a luminous disturbance localized in the immediate neighbourhood of* O *, the effect of it will be, for* $t = t'$*, localized in the immediate neighbourhood of the surface of the sphere with centre* O *and radius* $\omega t'$*: that is, will be localized in a very thin sperical shell with centre* O *including the aforesaid sphere"*
[*Jacques Hadamard: Lectures on Cauchy's problem in linear partial differential equations. 1923*]

Elementary Solution[1] : preliminaries

Let $(\mathcal{M}, g)$ be a n-dimensional pseudo-Riemannian manifold and P a normally hyperbolic operator

$$P\varphi \doteq \left(g^{\mu\nu}\nabla_\mu\nabla_\nu + a^\mu(x)\nabla_\mu + b(x)\right)\varphi = 0.$$

A set $\Omega \subset \mathcal{M}$ is called geodesically starshaped with respect to $x \in \Omega$ if there is an open subset $\Omega' \subset T_x\mathcal{M}$ which is starshaped with respect to $0 \in T_x\mathcal{M}$ such that $\exp_x : \Omega' \to \Omega$ is a diffeomorphism. We call a subset $\Omega \subset \mathcal{M}$ geodesically convex if it is geodesically starshaped with respect to all its points.



For any two points $x, y \in \Omega$, we can therefore define the squared geodesic distance $\sigma(x, y)$ as

$$\sigma(x, y) \doteq g(\exp_x^{-1}(y), \exp_x^{-1}(y)) \text{ which fulfils } g^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma = 4\sigma.$$

The *characteristic conoid* is $\sigma(x, y) = 0$: It is regular surface everywhere apart from the point x, the vertex of the conoid.

[1] Thanks to Jan-Hendrik for the suggestions!

Elementary Solution

The *elementary solution* in the sense of Hadamard for the hyperbolic equation is a function $\varphi : \Omega \times \Omega \to \mathbb{R}$ of the form:

$$\varphi(x,y) = \begin{cases} V(x,y)\sigma^{-p} + W_0(x,y)\log\sigma + \mathcal{S} & n \text{ even} \\ W_1(x,y)\sigma^{-p} & n \text{ odd} \end{cases}$$

where $p = (n-2)/2$, V, W_0, W_1 are smooth functions in a neighbourhood of the characteristic conoid, which admit in that neighbourhood the following expansion:

$$V(x,y) = \sum_{i=0}^{p-1} u_i(x,y)\sigma^i, \quad W_0(x,y) = \sum_{i=p}^{\infty} u_i(x,y)\sigma^{i-p},$$

$$W_1(x,y) = \sum_{i=0}^{\infty} u_i(x,y)\sigma^{i-p}$$

and $\mathcal{S}$ is a smooth function.

... and the fundamental solution?[2]

Nowadays a fundamental solution is usually taken to be a generalized (weak) solution belonging to the space $D'(\mathcal{M})$ of the equation

$$P\varphi_+ = \delta(x-y) \;\text{ with }\; supp(\varphi_+) \subset J^+(x).$$

In that case the elementary solution in a neighbourhood of the vertex of the characteristic conoid:

$$\varphi_+(x,y) = \sum_{i=0}^{\infty} \tilde{u}_i(x,y)\Sigma^{i-p}(\sigma)$$

where $\tilde{u}_i(x,y)$ are smooth functions that coincide up to a multiplication by a constant with the $u_i(x,y)$ and $\Sigma^\lambda(\sigma)$ are obtained by analytic continuation in the parameter λ from $\Sigma^\lambda(\sigma) = \sigma_+^\lambda/\Gamma(\lambda+1)$, $Re\lambda > -1$ where $\Gamma(\lambda+1)$ is the Euler gamma function.

... and in QFT?

[2] Some examples are available on "Wave equations on Lorentzian Manifolds and Quantization"

Wald's hypothesis

[Robert M. Wald: Trace anomaly of a conformally invariant quantum field in curved spacetime. Physical Review D (1978).]

"The formal expression for $T_{\mu\nu}$ is quadratic in the field and hence is intrinsically meaningless since the field operator is really a distribution; thus, renormalization is required. This problem is of considerable importance since a meaningful expression for the stress-energy operator is needed if one is to complete the theory (in the semiclassical approximation) by postulating the equation:

$$G_{\mu\nu} = 8\pi < T_{\mu\nu} >$$

to account for the back reaction effect of the quantum field on the gravitational field. [...] The point-separation prescription for renormalizing $T_{\mu\nu}$ is based on the assumption that the expectation value $<\varphi(x)\varphi(x') + \varphi(x')\varphi(x)>$ has the form of a Hadamard elementary solution."

Kay's conjecture

[*Giuseppe Gonnella and Bernard S. Kay: Can locally Hadamard quantum states have non-local singularities? Classical and Quantum Gravity (1989).*]

"Can a (quasi-free) locally Hadamard bi-distribution possess (spacelike) non-local singularities?"

YES $\rightarrow$ An example is a non-positive, weak bi-solution ω, which, while locally Hadamard, possesses non-local singularities.

Kay's conjecture: *"Suppose that ω is a quasi-free state satisfying the Klein-Gordon equation on a globally hyperbolic space-time. If the two-point distribution ω_2 has the usual commutator*

$$(\omega_2)_-(f \otimes g) = \imath\kappa(f \otimes g)$$

and positivity properties

$$\omega_2(f^* \otimes f) \geq 0$$

and is locally Hadamard, then ω_2 is globally Hadamard."

Radzikowski's solution

[*Marek J. Radzikowski: A Local-to-Global Singularity Theorem for Quantum Field Theory on Curved Space-Time. Communication in Mathematical Physics (1996).*]

$(x_0, k_0) \in \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$ is a *regular direction* of a distribution u if $\exists\, f \in C_0^\infty(\mathbb{R}^m)$ with $f(x_0) \neq 0$ and $\exists$ conic neighborhood V of k_0 such that, $\forall n \in \mathbb{N}\ \exists C_n \in \mathbb{R}$ fulfilling:

$$|\widehat{fu}(k)| \leq C_n(1 + |k|)^{-n} \ \text{ for all } k \in V.$$

The *wave front set* is defined to be

$$\mathrm{WF}(u) = \{(x,k) \in \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\}) \big| (x,k) \text{ is } \textit{not} \text{ regular directed for } u\}$$

.

Corollary 11.1: *" If $(\mathcal{M}, g)$ is a 4-dimensional globally hyperbolic curved space-time, and ω_2 is a locally Hadamard two-point distribution satisfying Klein-Gordon equation and commutator property mod C^∞, then the following statements are equivalent:*

(i) *ω_2 is globally Hadamard, namely*

$$WF(\omega_2) = \left\{(x, y, k_x, -k_y) \in T^*M^2 \setminus 0 \,\Big|\, (x, k_x) \sim (y, k_y), \quad k_x \triangleright 0 \right\},$$

(ii) *ω_2 is of positive type mod C^∞."*

Sahlmann and Verch's extension

[*Hanno Sahlmann and Rainer Verch: Microlocal spectrum condition and Hadamard form for vector-valued quantum fields in curved spacetime. Reviews in Mathematical Physics (2001).*]

Let be $\mathfrak{X}$ a C^∞ vector bundle over a base manifold $\mathcal{M}$ with typical fibre $\mathbb{C}^r$ and bundle projection $\pi_{\mathcal{M}}$. Then let $U \subset \mathcal{M}$ be an open subset and let $(e_1, \ldots, e_r)$ be a local trivialization of $\mathfrak{X}$ over U. Such a local trivialization induces a one-to-one correspondence between $C_0^\infty(\mathfrak{X}_U)$ and $\oplus^r \mathcal{D}(U)$ by assigning to each $f \in C_0^\infty(\mathfrak{X}_U)$ the $(f_1, \ldots, f_r) \in \oplus^r \mathcal{D}(U)$ with $f_a e^a = f$.
With this notation, one defines for $u \in (C_0^\infty(\mathfrak{X}_U))'$ the wavefront set as

$$\mathrm{WF}(u) \doteq \bigcup_{a=1}^{r} \mathrm{WF}(u_a).$$

Theorem 5.5 *Let $\omega \in (C_0^\infty(\mathfrak{X} \boxtimes \mathfrak{X}))'$ be a bi-solution mod C^∞ for the wave-operator P. Moreover, assume that there is a Cauchy-surface Σ in $(\mathcal{M}, g)$ having a causal normal neighbourhood N so that ω is of Hadamard form for the wave-operator on N. Then, if N' is a causal normal neighbourhood of any other Cauchy surface Σ' in $(\mathcal{M}, g)$, ω is also of Hadamard form for the wave-operator P on N'.*

Thank you for your attention!

Your questions, suggestions, comments
and remarks are welcome!

Example: Hadamard solution in cosmological spacetime

[Amir-Homayoon Najmi and Adrian C. Ottewill: Quantum states and the Hadamard form. I. Energy minimization for scalar fields. Physical Review D (1984).]

Consider an arbitrary spatially flat Robertson-Walker universe. The Hadamard ansatz postulates that in a geodesically convex neighbourhood we can write our state:

$$G_H^{(1)}(x,x') = \frac{1}{4\pi^2}\left[\frac{u}{\sigma} + v\ln\sigma + w\right]$$

where σ is one half the geodesic distance between x and x' and $u(x,x')$, $v(x,x')$, and $w(x,x')$ are smooth functions,

$$v(x,x') = \sum_{n=0}^{\infty} v_n(x,x')\sigma^n, \quad w(x,y) = \sum_{n=0}^{\infty} w_n(x,x')\sigma^n.$$

Imposing $(\Box - \xi R - m^2)G_H^{(1)} = 0$ determines $u(x,x') = \Delta^{\frac{1}{2}}(x,x')$, where $\Delta(x,x') = g^{-\frac{1}{2}}(x)\det(\sigma_{;\mu\nu})g^{-\frac{1}{2}}(x')$.

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The same equation also yields recursion relations for the coefficients v_n and w_n:

$$(n+1)(n+2)v_{n+1}+(n+1)v_{n+1;\mu}\sigma^{;\mu}+ \\ -(n+1)v_{n+1}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\sigma^{;\mu}+\frac{1}{2}(\Box-\xi R-m^2)v_n=0, \tag{1}$$

$$(n+1)(n+2)w_{n+1}+(n+1)w_{n+1;\mu}\sigma^{;\mu}+ \\ -(n+1)w_{n+1}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\sigma^{;\mu}+\frac{1}{2}(\Box+-\xi R-m^2)w_n+ \\ -(2n+3)v_{n+1}+v_{n+1;\mu}\sigma^{;\mu}-v_{n+1}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\sigma^{;\mu}=0. \tag{2}$$

Together with the boundary condition

$$v_0+v_{0;\mu}\sigma^{;\mu}-v_0\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\sigma^{;\mu}+\frac{1}{2}(\Box-\xi R-m^2)\Delta^{\frac{1}{2}}=0. \tag{3}$$

These equations can be solved by integrating along the geodesic from x' to x. The coefficients $v_n(x,x')$ are uniquely determined by (1) and (3). The coefficient $w_n(x,x')$ are determined once that $w_0(x,x')$ is specified. The indeterminacy corresponds to the freedom to add to $G_H^{(1)}(x,x')$ any non-singular symmetric solution to the wave equation.

Example: Hadamard solution in cosmological spacetime

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We can solve equation (3) by expanding $v_0(x, x')$ in a Taylor series and using

$$\Delta^{\frac{1}{2}} = 1 + O(\sigma), \quad \Box\Delta^{\frac{1}{2}} = \frac{1}{6}R + O(\sigma).$$

This yields the following expression

$$G_H^{(1)}(x, x') = \frac{1}{4\pi^2}\left\{\frac{1}{\sigma} + \frac{1}{2}\left[m^2 + (\xi - \frac{1}{6})R\right]\ln\sigma + O(1)\right\}.$$

The appropriate expansion can be obtained by solving the geodesic equation $\sigma^{;\mu}\sigma_{;\mu} = 2\sigma$ with the boundary conditions $\sigma_{,\mu}(x, x) = 0$ and $\sigma_{,\mu\nu}(x, x) = g_{\mu\nu}$

$$G_H^{(1)}(t_0, x; t_0, x') = \frac{1}{4\pi^2}\left\{\frac{2}{a_0^2 r^2} + \frac{1}{2}\left[m^2 + (6\xi - 1)\frac{\ddot{a}_0}{a_0^3}\right]\ln(r^2) + O(1)\right\}.$$

For a massless, conformally coupled field, we obtain

$$G_H^{(1)}(t, x; t', x') = \frac{1}{2\pi^2}\frac{a_0^{-1}(t)a_0^{-1}(t')}{-(-t - t')^2 + r^2}.$$

□

Example: Failure of positivity for a class of locally Hadamard weak bisolutions with non-local singularities

[*Giuseppe Gonnella and Bernard S. Kay: Can locally Hadamard quantum states have non-local singularities? Classical and Quantum Gravity (1989).*]

Consider the Klein-Gordon equation for a real scalar field in flat spacetime.
Let G_0 be a state and, for each $c \geq 1$, consider the (symmetric bilinear) map G_c on test functions defined by

$$G_c(f_1, f_2) = G_0(f_1, f_2) + cG_0(f_1^T, f_2)$$

with $f^T(f, x) = f(-t, x)$. For example, in the massless case

$$G_c(t_1, x_1; t_2, x_2) = \frac{1}{2\pi^2}[(t_1 - t_2)^2 - (x_1 - x_2)^2]^{-1} + \frac{c}{2\pi^2}[(t_1 + t_2)^2 - (x_1 - x_2)^2]^{-1}.$$

Because of its ‘anomalous behaviour’ on the hyperplane $t = 0$, it is not everywhere locally Hadamard on all of $(\mathcal{M}, \eta)$. However, by restricting to a suitable (globally hyperbolic) "sub"-spacetime such as a ‘time band’

$$\mathcal{M}_{(a,b)} = \{(t, x) \in \mathcal{M} : a < t < b\}$$

with $0 < a < b < \infty$ or with $-\infty < a < b < 0$, G_c, provides an example of an everywhere locally Hadamard weak bisolution.

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We shall prove that, for all $c \geq 1$, G_c, fails to satisfy the positivity conditions hence cannot be a state.

Consider an ('odd') test function f_0 which satisfies $f_0^T = -f_0$ and $\kappa * f_0 \neq 0$. Since the G_c has to satisfies

$$G_c(f_0, f_0) > 0, \quad \kappa * f_0 \neq 0, \quad \kappa(f_1, f_2)^2 \leq G_c(f_1, f_1) G_c(f_2, f_2)$$

in order to be a state, G_0 has to satisfies $G_0(f_0, f_0) > 0$.

On the other hand,

$$f_0^T = -f_0 \quad \Longrightarrow \quad G_c(f_0, f_0) = (1 - c) G_0(f_0, f_0) < 0$$

which contradicts $G_c(f_0, f_0) > 0$.

□