Invariant states on Weyl algebras for the action of the symplectic group

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Joint project with Federico Bambozzi and Nicola Pinamonti

PHYSICAL THEORY

measurment: (state, physical observable) $\mapsto$ results

ALGEBRAIC APPROACH

– Observables collected into a $*$-algebra $\mathcal{A}$

– Symmetries implemented as algebra automorphism

– States are positive normalized linear maps $\omega : \mathcal{A} \to \mathbb{C}$

– GNS theorem $(\omega, \mathcal{A}) \Longleftrightarrow \left(\mathscr{H}_\omega, \Psi_\omega, \pi_\omega\right)$

QUESTION: How many **invariant states** we can find?

♡ GOAL: **Classify $Sp(2g, \mathbb{Z})$-invariant states on Weyl algebras**

Outline of the Talk

- **Weyl algebras for $\mathbb{Z}^2$ and automorphism induced by $Sp(2,\mathbb{Z})$**

- **Classification of $Sp(2,\mathbb{Z})$-invariant states**

- **Outlook**

- Based on:
  Invariant states on Weyl algebras for the action of the symplectic group
  Federico Bambozzi , S.M., Nicola Pinamonti - (arXiv:1802.02487 [math.OA])

Weyl algebras for $\mathbb{Z}^2$ and automorphism induced by $Sp(2,\mathbb{Z})$

WEYL ALGEBRAS

- Fix $h \in \mathbb{R}$ s.t. $\hbar := h/2\pi \in \mathbb{R} \setminus \mathbb{Q}$
- Choose a skew-symmetric, bilinear map $\sigma : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$

$$\sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- Weyl $*$-algebra $\mathcal{A}$ is defined by

$$\mathcal{A} = \operatorname{span}_{\mathbb{C}}\{\mathbb{Z}^2 \ni m \mapsto W_m \in C^0(\mathbb{Z}^2, \mathbb{C}) \mid W_m W_n = e^{\imath h \sigma(m,n)} W(n+m)\,,\ (W_m)^* = W_{-m}\}$$

ORBITS OF THE SYMPLECTIC GROUP

- Symplectic group $Sp(2,\mathbb{Z})\big(\equiv Sl(2,\mathbb{Z})\big)$ acts on $\mathbb{Z}^2 \ni m \mapsto \Theta m$, with det Θ=1
- $Sp(2,\mathbb{Z}) \ni \Theta \longmapsto \Phi_\theta \in \mathrm{Aut}(\mathcal{A})$ by linearity: $\Phi_\Theta W_m = W_{\Theta m}$
- Set of fixed points $= \{(0,0) \in \mathbb{Z}^2\} \Longrightarrow$ the action of Φ_Θ is *ergodic* on $\mathcal{A}$

$$\Phi_\Theta\big(\lambda W_{(0,0)}\big) = \lambda W_{(0,0)} \qquad \text{for any } \lambda \in \mathbb{C}\,, \Theta \in Sp(2,\mathbb{Z})$$

$Sp(2,\mathbb{Z})$-invariant states

Definitions: A $Sp(2,\mathbb{Z})$-**invariant state** ω if for any $\Phi_\Theta \in \text{Aut}(\mathcal{A})$, $\Theta \in Sp(2,\mathbb{Z})$ it holds

- $\omega(\mathfrak{a}^*\mathfrak{a}) \geq 0$
- $\omega(1_\mathcal{A}) = 1$
- $\omega(\Phi_\Theta \mathfrak{a}) = \omega(\mathfrak{a})$

N.B.: To construct ω, it is enough to prescribe its values on the generators of $\mathcal{A}$

$$\omega(W_m) = \begin{cases} 1 & \text{if } m = (0,0) \\ p^{(m)} \in \mathbb{C} & \text{else} \end{cases}$$

for a sequence of values $p^{(m)}$ and then extend it by linearity to any $\mathfrak{a} \in \mathcal{A}$

Theorem

The only $Sp(2,\mathbb{Z})$-invariant state on $\mathcal{A}$ is the **trace state** τ defined by

$$\tau(W_m) = \begin{cases} 1 & \text{if } m = (0,0) \\ 0 & \text{else} \end{cases}$$

N.B.: τ is obviously invariant:

$$\tau(\Phi_\Theta W_m) = \tau(W_{\Theta m}) = \begin{cases} 1 & \text{if } m = (0,0) \\ 0 & \text{else} \end{cases}$$

Sketch of the proof I

Assume by contraddiction there exists $Sp(2,\mathbb{Z})$-invariant state ω (different from τ!)

$$\omega(W_\xi) = \begin{cases} 1 & \text{if } \xi = (0,0) \\ p^{(\xi)} \in \mathbb{C} & \text{else} \end{cases}$$

Goal: $Sp(2,\mathbb{Z})$-invariance $\iff p^{(\xi)} = 0$ for any $\xi \neq 0 \implies \omega \equiv \tau$

(1) By positivity and by $Sp(2,\mathbb{Z})$-invariance, we have

$$\underbrace{\overline{\omega(W_\xi)} = \omega(W_\xi^*)}_{\text{positivity of } \omega} = \omega(W_{-\xi}) = \underbrace{\omega(W_{-Id\xi}) = \omega(W_\xi)}_{Sp(2,Z)\text{-invariace of } \omega}$$

(2) Choosing $\mathfrak{a} = W_0 + W_\xi \xRightarrow{\omega(\mathfrak{a}^*\mathfrak{a})\geq 0} 1 - p^2 \leq 0$

(3) Hence, any $Sp(2,\mathbb{Z})$-invariant states reads as

$$\omega(W_\xi) = \begin{cases} 1 & \text{if } \xi = (0,0) \\ p^{(\xi)} \in [-1,1] & \text{else} \end{cases}$$

Sketch of the proof II

(4) Let $m, n \in \mathbb{N}$ s.t. $\frac{m}{n} \in \mathbb{N}$ and $n > 1$ and consider $\mathcal{V}_{d+1,n} \subset \mathcal{A}$ with elements of the form

$$\mathfrak{a} = \alpha_0 W_{(0,0)} + \sum_{j=1}^{d} \alpha_j W_{\Theta_j \xi} \qquad \text{with } \Theta_j := \begin{pmatrix} 1 + \frac{m}{n}(n-1)j & \frac{m}{n}j \\ n-1 & 1 \end{pmatrix} \in Sp(2, \mathbb{Z})$$

(5) For any $\mathcal{V}_{d+1,n}$, the map $\mathfrak{a} \mapsto \omega(\mathfrak{a}^* \mathfrak{a})$ is a quadratic form $\omega(\mathfrak{a}^* \mathfrak{a}) = \overline{\alpha}^t \, \mathbf{H}_n \, \alpha$

(6) The set of positive Hermitian matrices form a convex cone

$$\mathbf{P}_d = \sum_{n=1}^{d} \frac{1}{d} \mathbf{H}_n \geq 0$$

(7) For fixed p exists d "big enough" s.t. $\det(\mathbf{P}_d) = 1 - dp^2 < 0 \Rightarrow \mathbf{P}_n \not\geq 0 \Rightarrow \mathbf{H}_n \not\geq 0$

(8) Hence $p = 0$ is a <u>necessary</u> condition for ω being an $Sp(2, \mathbb{Z})$-invariant state

□

Outlook

WHAT WE DID:

- Weyl $*$-algebra $\mathcal{A}$ constructed for $\mathbb{Z}^{2g}$
- unique $Sp(2g, \mathbb{Z})$-invariant state on $\mathcal{A}$ and it is the tracial state

$$\tau(\Phi_\Theta W_m) = \tau(W_{\theta m}) = \begin{cases} 1 & \text{if } m = (0,0) \\ 0 & \text{else} \end{cases}$$

WHAT COMES NEXT?

- assign a Weyl algebra to $(\mathcal{G}, Sp)$ with generic abelian group $\mathcal{G}$
- classify Sp-invariant states: $\diamond$ $\mathcal{G}$ is torsion free $\diamond$ $\mathcal{G}$ has nontrivial torsion subgroup

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THANKS for your attention!