Is there a ‘natural state’ for Abelian Chern-Simons theory?

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FOUNDATIONAL AND STRUCTURAL ASPECTS OF GAUGE THEORIES

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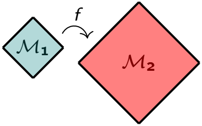
Motivation

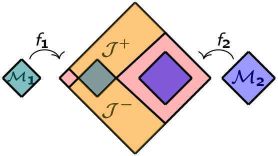
Locally covariant QFT: Loc $\to$ Alg

✓ **Isotony**



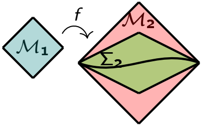
$\mathfrak{A}(\mathcal{M}_1) \xrightarrow{\text{mono}} \mathfrak{A}(\mathcal{M}_2)$

✓ **Causality**



$[\mathfrak{A}(\mathcal{M}_1), \mathfrak{A}(\mathcal{M}_2)] = \{0\}$

✓ **Time-Slice axiom**



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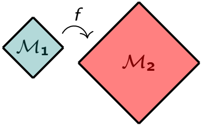

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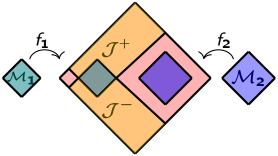

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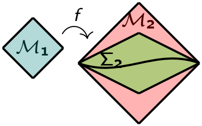

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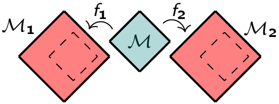
¿ Natural state ?



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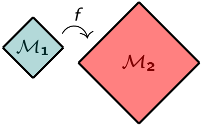

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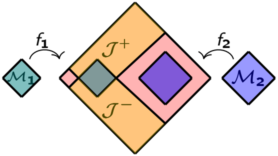

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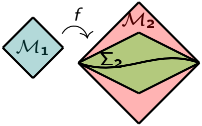

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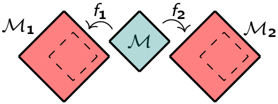

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No-Go Theorem for dynamicalQFT:
Fewster and Verch (2012)

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Locally covariant QFT: Loc $\to$ Alg

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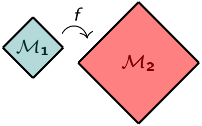

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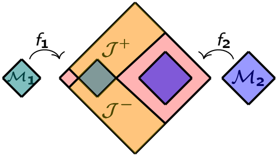

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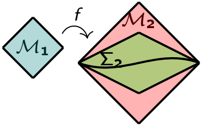

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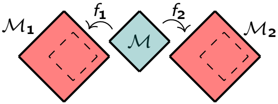

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No-Go Theorem for dynamicalQFT:
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GOAL: **investigate 'natural state' in Topological QFT**

- **Abelian Chern-Simons theory**
- **Quantization in the algebraic approach**
- **Invariant functionals on compact surfaces**
- **¿ Natural states ?**

Based on:

- C. Dappiaggi, S.M., A. Schenkel - Journal of Geometry and Physics 116 (2017).

The moduli space of flat $U(1)$-connections

- We consider $\mathcal{M} \simeq \mathbb{R} \times \Sigma$ with Σ an oriented surface
- Abelian Chern–Simons theory is characterized by

$$S = \frac{1}{4\pi} \int_{\mathcal{M}} A \wedge dA \quad \text{with} \quad A \in \Omega^1(\mathcal{M})$$

  and the solution space of the Euler-Lagrange equation is

$$\frac{\delta S}{\delta A} = \frac{1}{2\pi} dA = 0 \quad \Longrightarrow \quad \mathcal{S}\mathrm{ol}(\mathcal{M}) = \Omega^1_d(\mathcal{M})$$

- Taking the quotient by gauge transformations

$$A \mapsto A + \frac{1}{2\pi\imath} d \log g \quad \text{with} \quad g \in C^\infty(\mathcal{M}, U(1))$$

  we obtain the gauge space $\dfrac{\Omega^1_d(M)}{\Omega^1_{\mathbb{Z}}(M)}$

- The moduli space of flat $U(1)$-connection:

$$\mathrm{Flat}_{U(1)} := \frac{\Omega^1_d(\mathcal{M})}{\Omega^1_{\mathbb{Z}}(\mathcal{M})} \simeq \frac{H^1(\mathcal{M};\mathbb{R})}{H^1(\mathcal{M};\mathbb{Z})} \simeq \frac{H^1(\Sigma;\mathbb{R})}{H^1(\Sigma;\mathbb{Z})} \simeq \frac{\Omega^1_d(\Sigma)}{\Omega^1_{\mathbb{Z}}(\Sigma)}$$

Functoriality

- Consider the categories

$$\mathsf{Man}_2 = \begin{cases} \mathsf{Obj}(\mathsf{Man}_2) = \{\text{2-dimensional oriented manifolds}\} \\ \mathsf{Mor}(\mathsf{Man}_2) = \{\text{orientation preserving open embeddings}\} \end{cases}$$

$$\mathsf{Ab} = \begin{cases} \mathsf{Obj}(\mathsf{Ab}) = \{\text{Abelian groups}\} \\ \mathsf{Mor}(\mathsf{Ab}) = \{\text{group homomorphisms}\} \end{cases}$$

- The assignment of the moduli spaces is a functor

$$\mathsf{Flat}_{U(1)} : \mathsf{Man}_2^{\mathsf{op}} \longrightarrow \mathsf{Ab}$$

which assigns to a morphism $f : \Sigma \to \Sigma'$ in Man_2 the corresponding

$$\mathsf{Flat}_{U(1)}(f) := f^* : \frac{\Omega^1_d(\Sigma')}{\Omega^1_{\mathbb{Z}}(\Sigma')} \longrightarrow \frac{\Omega^1_d(\Sigma)}{\Omega^1_{\mathbb{Z}}(\Sigma)} \qquad [A'] \longmapsto [f^* A']$$

Observables for Abelian Chern-Simons theory

- As basic osservables we take all group characters on $\mathrm{Flat}_{U(1)}$

$$\mathrm{Hom}\Big(\mathrm{Flat}_{U(1)}(\Sigma), U(1)\Big)$$

- In terms of compactly supported 1-form, we define the group characters on $\Omega^1_d(\Sigma)$

$$\Omega^1_d(\Sigma) \longmapsto U(1)\,, \qquad A \mapsto \exp\Big(2\pi\imath \int_\Sigma \varphi \wedge A\Big)$$

- This character descends to the quotient if and only if

$$\int_\Sigma \varphi \wedge \Omega^1_{\mathbb{Z}}(\Sigma) \subseteq \mathbb{Z}$$

- Since $d\Omega^0(\Sigma) \subseteq \Omega^1_{\mathbb{Z}}(\Sigma)$, Stokes' lemma implies that $\varphi \in \Omega^1_{c,d}(\Sigma)$

- Because each exact $\varphi = d\chi \in d\Omega^0_c(\Sigma)$ yields a trivial group character

$$H^1_c(\Sigma;\mathbb{Z}) := \Big\{[\varphi] \in H^1_c(\Sigma;\mathbb{R}) : \int_\Sigma \varphi \wedge H^1(\Sigma) \subseteq \mathbb{Z}\Big\}$$

Functoriality

- The assignment of the character groups is a functor

$$H^1_c(-;\mathbb{Z}) : \mathsf{Man}_2 \to \mathsf{Ab}$$

which assigns to a morphism $f : \Sigma \to \Sigma'$ in Man_2 the corresponding

$$H^1_c(f;\mathbb{Z}) := f_* \ : \ H^1_c(\Sigma;\mathbb{Z}) \longrightarrow H^1_c(\Sigma';\mathbb{Z}) \,, \quad [\varphi] \longmapsto [f_*\varphi]$$

- $H^1_c(\Sigma;\mathbb{Z})$ can be equipped with a presymplectic structure

$$\tau_\Sigma : H^1_c(\Sigma;\mathbb{Z}) \times H^1_c(\Sigma;\mathbb{Z}) \longrightarrow \mathbb{R} \qquad \tau([\varphi],[\widetilde{\varphi}])_\Sigma = \int_\Sigma \varphi \wedge \widetilde{\varphi}$$

- Since any morphism $f : \Sigma \to \Sigma'$ in Man_2

$$\tau_{\Sigma'}\big(f_*[\varphi], f_*[\widetilde{\varphi}]\big) = \int_{\Sigma'} (f_*\varphi) \wedge (f_*\widetilde{\varphi}) = \int_\Sigma \varphi \wedge (f^* f_* \widetilde{\varphi}) = \int_\Sigma \varphi \wedge \widetilde{\varphi} = \tau_\Sigma\big([\varphi],[\widetilde{\varphi}]\big) \ ,$$

the assignment of the character groups can be promoted to be a functor

$$\mathcal{O} = \Big(H^1_c(-;\mathbb{Z}), \tau\Big) : \mathsf{Man}_2 \to \mathsf{PAb}$$

Quantization of Abelian Chern-Simons theory I

- Quantization is achieved as the functor $\mathcal{A} := \mathfrak{CCR} \circ \mathcal{O} : \mathsf{Man}_2 \to \mathsf{CAlg}$

1) We construct a $\mathbb{C}$-vector space $\Delta := \mathrm{span}_{\mathbb{C}} \{ W_{[\varphi]} \mid [\varphi] \in H^1_c(\Sigma; \mathbb{Z}) \}$

2) We define the structure of an associative unital $*$-algebra on Δ by

   $\diamond$ ${W_{[\varphi]}}^* := W_{-[\varphi]}$ $\quad\diamond$ $W_{[\varphi]}\, W_{[\widetilde{\varphi}]} := e^{-i\hbar\, \tau_\Sigma([\varphi],[\widetilde{\varphi}])}\, W_{[\varphi]+[\widetilde{\varphi}]}$ $\quad\diamond$ $\hbar \notin 2\pi\mathbb{Z}$

3) Given any Man_2-morphism, we construct a $*$-algebra homomorphism

$$\mathcal{A}(f) : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \qquad \mathcal{A}(f)\big(W_{[\,\cdot\,]}\big) := W'_{f_*[\,\cdot\,]}$$

4) We complete Δ with respect to the $*$-norm $\| \cdot \|^{\mathsf{Ban}}$

$$\| \sum \alpha_i W_{[\varphi]_i} \|^{\mathsf{Ban}} = \sum |\alpha_i|$$

Quantization of Abelian Chern-Simons theory II

5) On Δ^{Ban} there exists a faithful state, i.e. $\omega(a^* a) \geq 0$ and $\omega(W_0) = 1$,

$$tr\left(W_{[\varphi]}\right) := \begin{cases} 1 & \text{if } [\varphi] = 0 \\ 0 & \text{else} \end{cases}$$

6) We obtain a C^*-algebra taking the completion of Δ with respect to the norm

$$||a||^{m.r.n.} := \sup_{\omega \in \mathcal{F}} \sqrt{\omega(a^* a)}$$

where $\mathcal{F}$ is the set of states on Δ^{Ban}

Remark: Every normalized, linear functional ω can be written as

$$\omega\left(W_{[\varphi]}\right) := \begin{cases} 1 & \text{if } [\varphi] = 0 \\ K_{[\varphi]} & \text{else} \end{cases}$$

Invariant functionals on compact surfaces I

- Any object in Man_2 comes together with its automorphisms $\mathsf{Diff}^+(\Sigma)$ of Σ
- The functor $\mathcal{A} : \mathsf{Man}_2 \to \mathsf{CAlg}$ defines a representation

$$\mathsf{Diff}^+(\Sigma) \longrightarrow \mathsf{Aut}\big(\mathcal{A}(\Sigma)\big) \tag{1}$$

- Because of $\mathcal{A}(f)\big(W_{[\,\cdot\,]}\big) := W'_{f_*[\,\cdot\,]}$ and the fact that $H^1_c(\Sigma;\mathbb{Z})$ is discrete

$$\mathsf{Diff}^+_0(\Sigma) \subseteq \mathsf{Diff}^+(\Sigma) \text{ is represented trivially}$$

- The representation (1) descends to a representation of the mapping class group

$$\mathsf{MCG}(\Sigma) := \frac{\mathsf{Diff}^+(\Sigma)}{\mathsf{Diff}^+_0(\Sigma)} \to \mathsf{Aut}\big(\mathcal{A}(\Sigma)\big) \qquad [f] \mapsto \mathcal{A}(f)$$

- For compact Σ, there exists a short exact sequence of groups

$$1 \longrightarrow \mathsf{Tor}(\Sigma) \longrightarrow \mathsf{MCG}(\Sigma) \longrightarrow \mathsf{Sp}(H^1(\Sigma;\mathbb{Z}),\tau_\Sigma) \longrightarrow 1$$

Invariant functionals on compact surfaces II

- The representation of the MCG(Σ) descends to a representation of $\mathrm{Sp}(H^1(\Sigma;\mathbb{Z}),\tau_\Sigma)$

$$\mathrm{Sp}(H^1(\Sigma;\mathbb{Z}),\tau_\Sigma) \longrightarrow \mathrm{Aut}(\mathcal{A}(\Sigma)) \ , \quad T \longmapsto \kappa_T : W_{[\varphi]} \mapsto W_{T[\varphi]}$$

- An invariant functional under the action of the symplectic group

$$\omega\big(W_{T[\varphi]}\big) = \omega\big(W_{[\varphi]}\big) := \begin{cases} 1 & \text{if } [\varphi] = 0 \\ K_{[\varphi]} & \text{else} \end{cases}$$

Remark 1: The tracial state $\Big(K_{[\varphi]} = 0 \text{ for every } \varphi\Big)$ is clearly invariant

Remark 2: Since $\mathrm{Sp}(H^1(\Sigma;\mathbb{Z}),\tau_\Sigma) \not\subset \mathrm{O}(H^1(\Sigma;\mathbb{Z}),\mu)$, do *not* exist invariant Gaussian states

$$\omega\big(W_{[\varphi]}\big) = e^{-\mu([\varphi],[\varphi])}$$

Question: Is the tracial state the only invariant state?

Non-existence of natural states

No-go theorem: There exist *no natural states* on the functor $\mathcal{A} : \mathsf{Man}_2 \to \mathsf{CAlg}$, namely a state for each Σ such that for all Man_2-morphisms $f : \Sigma \to \Sigma'$ holds true:

$$\omega_{\Sigma'} \circ \mathcal{A}(f) = \omega_\Sigma$$

Sketch of the proof

- Let us assume that there exists a natural state $\{\omega_\Sigma\}_{\Sigma \in \mathsf{Man}}$
- Consider the Man_2-diagram: $\mathbb{S}^2 \xleftarrow{f_1} \mathbb{R} \times \mathbb{S} \xrightarrow{f_2} \mathbb{T}^2$
- The naturality of the state implies: $\omega_{\mathbb{S}^2} \circ \mathcal{A}(f_1) = \omega_{\mathbb{R}\times\mathbb{S}} = \omega_{\mathbb{T}^2} \circ \mathcal{A}(f_2)$
- Because of $H^1(\mathbb{S}^2;\mathbb{Z}) = 0$, then $\mathcal{A}(\mathbb{S}^2) \simeq \mathbb{C}$ and hence $\omega_{\mathbb{S}^2} = \mathrm{id}_\mathbb{C}$ is unique on $\mathbb{C}$
- We can choose f_2 such that $W_n^{\mathbb{R}\times\mathbb{S}} \mapsto W_{(n,0)}^{\mathbb{T}^2}$ we obtain that $\omega_{\mathbb{T}^2}\big(W_{(n,0)}^{\mathbb{T}^2}\big) = 1$
- Choosing $a = \alpha_1\, 1 + \alpha_2\, W_{(1,1)}^{\mathbb{T}^2} + \alpha_3\, W_{(0,1)}^{\mathbb{T}^2} \in \mathcal{A}(\mathbb{T}^2)$ the functional $\omega_{\mathbb{T}^2}(a^* a) < 0$

Q.E.D

Conclusions

- The C-S functor $\mathcal{A} : \mathsf{Man}_2 \to \mathsf{CAlg}$ violates the *isotony axiom* of LCQFT

$$\mathcal{A}(f) : \mathcal{A}(\mathbb{R} \times \mathbb{S}) \to \mathcal{A}(\mathbb{S}^2) \text{ is not injective}$$

- Violation of isotony seems to be a general feature of quantum gauge theories [Becker, Benini, Dappiaggi, Hack, Sanders, Schenkel, Szabo, . . .]

$$\Downarrow$$

"Future prospective": Possible way to solve the violation of isotony is

homotopical LCQFT := homotopical algebra + LCQFT

Questions: (1) Can we define natural states for homotopical LCQFT?
(2) Is the tracial state the only invariant state?