HADAMARD STATES FOR MAXWELL THEORY BY PSEUDODIFFERENTIAL CALCULUS[1]

Simone Murro

Department of Mathematics
University of Genoa

Heriot-Watt Analysis Seminar

October 10, 2023

[1]joint project with Gabriel Schmid, Ph.D. student in Genoa

SETTING

- *Spacetime* is a globally hyperbolic manifold:

$$\mathrm{M} = \mathbb{R} \times \Sigma \qquad g = -\beta^2 dt^2 + h_t$$

- *Maxwell fields* are $A \in \Omega^1(\mathrm{M})$ subordinated to

$$PA = \delta dA = 0$$

DIFFICULTIES

- The operator P is 'hyperbolic' modulo a gauge transformation

$$A \mapsto A' = A + df \implies PA = 0 \Leftrightarrow \begin{cases} \Box A' = 0 \\ \delta A' = 0 \end{cases}$$

HOW CAN WE QUANTIZE IT?

$\hookrightarrow$ what does it means "to quantize a theory"?

The algebraic approach to quantum field theory I/II

- Let $M = \mathbb{R} \times \Sigma$ be globally hyperbolic
- Let $\varphi \in C^\infty(\mathrm{M})$ be (complex) scalar field satisfying

$$P\varphi = (-\Box + m^2)\varphi = 0$$

CLASSICAL THEORY

- The *Cauchy problem* is well-posed:

$$C^\infty_c(\Sigma) \oplus C^\infty_c(\Sigma) =: \mathcal{V}_\Sigma \simeq ker\ P$$

- There exists *Green operators* $\mathrm{G}^\pm : C^\infty_c(M) \to C^\infty(M)$

$$\mathrm{G}^\pm P|_{C^\infty_c(M)} = P\mathrm{G}^\pm = Id \qquad \mathrm{supp}(G^\pm f) \subset J^\pm(\mathrm{supp} f)$$

- *Phase space* is characterized using the causal propagator $\mathrm{G} := \mathrm{G}^+ - \mathrm{G}^-$

$$\mathcal{V}_\mathrm{P} := \frac{C^\infty_c(\mathrm{M})}{PC^\infty_c(\mathrm{M})} \simeq ker\ P$$

The algebraic approach to quantum field theory II/II

The phase space comes together with a *charge*, i.e. Hermitian form

$$\mathrm{q}(g,f) = (g, \mathrm{i}\mathrm{G}f) := \int_M \overline{g}(x)(\mathrm{i}\mathrm{G}f)(x)\mathrm{vol}_g$$

QUANTUM THEORY

Step 1: assign to any $v \in \mathcal{V}_\mathrm{P}$ an abstract element of the algebra $\mathrm{CCR}(\mathcal{V}_\mathrm{P}, \mathrm{q})$

generators: $\Phi(v) \qquad \Phi^*(v) \qquad \mathbb{1}$

CCR relations:
$$[\Phi(v), \Phi(w)] = [\Phi^*(v), \Phi^*(w)] = 0$$
$$[\Phi(v), \Phi^*(w)] = \mathrm{q}(v,w)\mathbb{1}$$

Step 2: Construct an **Hadamard states** $\omega : \mathrm{CCR}(\mathcal{V}_\mathrm{P}, \mathrm{q}) \to \mathbb{C}$ defined by

covariances: $\Lambda^+(v,w) := \omega(\Phi(v)\Phi^*(w)) \qquad \Lambda^-(v,w) := \omega(\Phi^*(w)\Phi(v))$

Hadamard conditions: $\mathrm{WF}'(\Lambda^\pm) \subset \mathcal{N}^\pm \times \mathcal{N}^\pm \qquad$ where: $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$

$\hookrightarrow$ Physical admissibility $\Longleftrightarrow$ microlocal analysis

DEFINITION: Let $u \in D'(M)$ be a distribution. We call

- **singular support** of u

$$singsupp(u) = \{p \in \mathrm{M} \mid \nexists O \ni p \text{ such that } u|_O \in C^\infty(O)\}.$$

- **wavefront set** of u

$$WF(u) = \{(p,k) \in T^*M \setminus \{0\} \mid p \in singsupp(u) \text{ and } k \in \Sigma_p(u)\},$$

where $\Sigma_p(u) = \cap_\rho \Sigma(\rho u)$ with $\rho(p) \neq 0$ and

$$\Sigma(\rho u) = \{k \in \mathrm{R}^n \setminus \{0\} \mid \nexists \text{ a conic } V \ni k \text{ such that } |\widehat{\rho u}|(k') \leq C_N(1 + |k'|)^{-N}, \forall N \in \mathrm{N} \text{ and } \forall k' \in V\}.$$

EXAMPLE: Dirac delta distribution $\delta(x)$:

$$\begin{cases} singsupp(\delta) = \{0\} \\ \widehat{(\rho\delta)}(k) = \rho(0) \end{cases} \implies WF(\delta) = \{(0,k)\}$$

DEFINITION: Let $u \in D'(M)$ be a distribution. We call

- **singular support** of u

$$singsupp(u) = \{p \in \mathrm{M} \mid \nexists O \ni p \text{ such that } u|_O \in C^\infty(O)\}.$$

- **wavefront set** of u

$$WF(u) = \{(p,k) \in T^*M \setminus \{0\} \mid p \in singsupp(u) \text{ and } k \in \Sigma_p(u)\},$$

where $\Sigma_p(u) = \cap_\rho \Sigma(\rho u)$ with $\rho(p) \neq 0$ and

$$\Sigma(\rho u) = \{k \in \mathrm{R}^n \setminus \{0\} \mid \nexists \text{ a conic } V \ni k \text{ such that}$$
$$|\widehat{\rho u}|(k') \leq C_N(1+|k'|)^{-N}, \forall N \in \mathrm{N} \text{ and } \forall k' \in V\}.$$

EXAMPLE: Covariance of an Hadamard state $\Lambda^\pm$

$$WF(\Lambda^\pm) = \{(x,k_x,y,k_y) \in T^*M \times T^*M \setminus \{0\} \mid (x,k_x) \sim (y,-k_y), \pm k_x \rhd 0\}$$
$$WF'(\Lambda^\pm) := \{(x,k_x,y,-k_y) \in T^*M \times T^*M \setminus \{0\} \mid (x,k_x,y,k_y) \in WF(\Lambda^\pm)\}$$

RECIPE FOR CONSTRUCTING HADAMARD STATES

0) For <u>simplicity</u> we work on *ultrastatic spacetimes*, i.e. $g = -dt^2 + h$

1) Replace the phase space $(\mathcal{V}_{\mathrm{P}}, \mathrm{q})$ with the space of initial data $(\mathcal{V}_\Sigma, \mathrm{q}_\Sigma)$

$$\rho G : (\mathcal{V}_{\mathrm{P}}, \mathrm{q}) \xrightarrow[\text{unitary}]{\simeq} (\mathcal{V}_\Sigma, \mathrm{q}_\Sigma) \qquad \mathrm{q}_\Sigma(\cdot,\cdot) := (\cdot\,, \mathrm{i}\mathsf{G}_\Sigma \cdot) \quad \mathsf{G} = (\rho G)^* \mathsf{G}_\Sigma (\rho G)$$

2) Construct an 'approximate' *square root of the (positive) Laplacian*:

$$\varepsilon^* = \varepsilon \qquad \varepsilon^{-1}\varepsilon = \mathbb{1} \qquad \varepsilon^2 = \Delta + r_{-\infty} \quad (\Psi\text{DO–calculus})$$

$$\Updownarrow \qquad \searrow$$

microlocal factorization of $\quad \Box = (\partial_t + \mathrm{i}\varepsilon)(\partial_t - \mathrm{i}\varepsilon) - r_{-\infty}$ (smoothing op.)

$$\Updownarrow \qquad \pi^\pm := \frac{1}{2}\begin{pmatrix} \mathbb{1} & \pm\varepsilon^{-1} \\ \pm\varepsilon & \mathbb{1} \end{pmatrix}$$

microlocal factorization of $\quad U_\Box = U_{(\partial_t + \mathrm{i}\varepsilon)}\pi^+ + U_{(\partial_t - \mathrm{i}\varepsilon)}\pi^-$

THEOREM [Gérard,Wrochna]: $\quad \Lambda^\pm(f,g) := (f, \lambda^\pm g)\ ,\ \lambda^\pm := \pm\mathrm{i}^{-1} U_\Box \pi^\pm \circ (\rho \mathsf{G})$

are pseudo-covariances for a quasifree Hadamard state $\omega : \mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \mathrm{q}) \to \mathbb{C}$.

INTERMEZZO II: pseudodifferential calculus

The *differential* operator $d/dx : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ can be written as

$$\frac{d}{dx} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} k \hat{f}(k)\, dk$$

hence a m-order differential operator A can be written as

$$Pf(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} p(x,k) \hat{f}(k) dk \qquad p(x,k) = \sum_{\alpha \leq m} a_\alpha(x) k^\alpha$$

The Kohn-Nirenberg quantization is the natural generalization

$$S^m_{1,0} \ni p(x,k) \mapsto P\Big(x, \frac{d}{dx}\Big) := Op(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ik(x-y)} p(x,k) f(y) dy\, dk \in \Psi^m(\mathbb{R})$$

where the symbol $p(x,k)$ is promoted to a smooth function in the class

$$S^m_{1,0} := \left\{ p \in C^\infty(\mathbb{R} \times \mathbb{R}) \,\middle|\, \left| \frac{d^\alpha}{dx^\alpha} \frac{d^\beta}{dk^\beta} (p(x,k)) \right| \leq C_{\alpha\beta} \langle k \rangle^{m-|\beta|} \;\; \forall \alpha, \beta \in \mathbb{N} \right\}$$

INTERMEZZO II: pseudodifferential calculus

NICE PROPERTIES:

- The ΨDO–calculus transforms covariantly under local diffeomorphisms:
  - $\psi : \mathbb{R}^n \to \mathbb{R}^n$ differomorphism
  - $U_i \subset \mathbb{R}^n$ precompact open sets and $\chi_i \in C_c^\infty(\mathbb{R}^n)$ s.t. $\chi_i|_{U_i} = 1$

  $\Rightarrow$ For $A \in \Psi^m(U_1)$ we have $\chi_1 A \psi^*(\chi_2 u) = Bu \in \Psi^m(U_2)$

  $\Rightarrow$ the definition of ΨDO extends on smooth manifolds

- Let $S^{-\infty} := \cap_m S^m_{1,0}$ and $\Psi^{-\infty}(M)$ accordingly:

  $\Rightarrow$ $A : D'(M) \to C^\infty(M)$ is *smoothing* if and only if $A \in \Psi^{-\infty}(M)$

  $\Rightarrow$ $WF(Au) = \emptyset$ for any $u \in \mathrm{D}'(M)$

- If M **compact** and $A \in \Psi^m(M)$ and $B \in \Psi^n(M)$

  $\Rightarrow$ $A \circ B \in \Psi^{m+n}$

  $\Rightarrow$ For polyhomogeneous symbols i.e. $\sigma_P \sim \sum_j \alpha_j k^j \Rightarrow \sigma_{AB} = \sigma_A \circ \sigma_B \in S^{m+n}_{ph}$

The ΨDO–calculus can be extended on *manifolds of bounded geometry*

CONSTRUCTION OF AN 'APPROXIMATE' SQUARE ROOT OF THE LAPLACIAN

(sketch of the proof)

- Let $M = \mathbb{R} \times \Sigma$ with Σ of bounded geometry
- The closure of the Laplacian $\overline{\Delta}$ with domain $H^2(\Sigma)$ is self-adjoint on $L^2(\Sigma)$
- We fix $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(0) = 1$ and set $\chi_R(\lambda) = \chi(R^{-1}\lambda)$ for $R \geq 1$
- We get $\chi_R(\overline{\Delta}) \in \Psi^{-\infty}(\Sigma)$ and we set $r_{-\infty} = R\chi_R(\overline{\Delta})$
- By the spectral calculus we find $R > 1$ s. t. $\overline{\Delta} + r-\infty$ is m-accreative
- By standard results of Kato, $\overline{\Delta} + r-\infty$ has a unique m-accreative square root

$$\varepsilon = \varepsilon^* \qquad \exists! \varepsilon^{-1} \in \Psi^{-1} \qquad \varepsilon^2 = \Delta + r_{-\infty}$$

□

HOW TO QUANTIZE MAXWELL'S THEORY?

$\hookrightarrow$ as a gauge theory

Step 1: Construct the classical phase space

$$
\begin{array}{ccc}
\mathrm{i}(\cdot, G_1 \cdot)_{V_1} =: q_1 \,,\ \mathcal{V}_P := \dfrac{\ker(\delta)}{\mathrm{ran}(P)} & \xrightarrow{\ [G_1]\ } & \dfrac{\ker(P)}{\mathrm{ran}(d)} \\
\text{unitary} \Big\downarrow [\rho_1 G_1] & \overset{[G_1]}{\searrow} & \Big\uparrow [j] \\
\mathrm{i}(\cdot, G_{1_\Sigma} \cdot)_{V_{\rho_1}} =: q_{1_\Sigma} \,,\ \mathcal{V}_\Sigma := \dfrac{\ker(K_\Sigma^\dagger)}{\mathrm{ran}(K_\Sigma)} & \xrightarrow{\ [\mathcal{U}_1]\ } & \dfrac{\ker(D_1) \cap \ker(\delta)}{d(\ker(D_0))}
\end{array}
$$

where

$$(\cdot, \cdot)_{V_1} := \int_M g^{-1}(\bar{\cdot}, \cdot)\, \mathrm{vol}_g$$

$P =: \delta d$ (dynamics) $\quad d$ (gauge freedom) $\quad \delta$ (constraint)

$D_1 := \delta d + d\delta \quad$ and $\quad D_0 = \delta d$ (auxiliary theories)

K_Σ (gauge freedom for initial data) $\quad K_\Sigma^\dagger$ (constrained initial data)

HOW TO CONSTRUCT HADAMARD STATES FOR A GAUGE THEORY?

DIFFERENCES WITH SCALAR CASE

- The fiber metric on $\mathcal{V}_\Sigma$ is not positive definite: $g^{-1}(\cdot,\cdot)$ is Lorentzian
- ΨDO–calculus interact badly with gauge invariance $c^{\pm} : (\mathrm{ran} K_\Sigma) \circlearrowleft$

$\hookrightarrow$ we fix all gauge-degrees of freedom

DEFINITION: $A = A_0 dt + A_\Sigma$ satisfies **Cauchy radiation gauge** on a Σ if

$$\delta A = 0 \ \textit{(Lorenz gauge)} \quad \text{and} \quad A_0|_\Sigma = \partial_t A_0|_\Sigma = 0$$

REMARK: On ultrastatic spacetimes, the following gauge are equivalent:

(i) A satisfies the *Cauchy radiation gauge*;

(ii) A satisfies the *temporal gauge* $A_0 = 0$ and the *Coulomb gauge* $\delta_\Sigma A_\Sigma = 0$;

(iii) The fiber metric g^{-1} reduces to h^{-1} in the Cauchy radiation gauge

$$g^{-1}(A,A) = -(A_0,A_0) + h^{-1}(A_\Sigma,A_\Sigma) = h^{-1}(A_\Sigma,A_\Sigma) \geq 0$$

THEOREM [M.-Schmid]: Let (Σ, h) be **complete** and define the Sobolev space

$$\mathcal{H}^s(\Sigma) = \mathrm{dom}(E^s) \qquad \text{where} \quad E := \overline{\Delta} + 1 \quad (\cdot,\cdot)_{H^s} := (E^{\frac{s}{2}}\cdot, E^{\frac{s}{2}}\cdot)_{L^2}$$

$\Longrightarrow$ the following is a $\mathcal{H}^s$-orthogonal Hodge-type decomposition

$$\mathcal{H}^s_k(\mathrm{M}) \cong \mathrm{Har}_k(\mathrm{M}) \oplus \overline{\mathrm{ran}(\overline{\mathrm{d}})} \oplus \overline{\mathrm{ran}(\overline{\delta})}$$

and any element in $\overline{\mathrm{ran}(\overline{\mathrm{d}})} \cap \Omega^k(\Sigma)$ is exact

COROLLARY: $\forall\, A \in \Omega^1(\mathrm{M})$ with $A|_\Sigma \in \mathcal{H}^s(\Sigma)$ $\exists\, f \in \Omega^0(\mathrm{M})$ with $f|_\Sigma \in \mathcal{H}^s(\Sigma)$ such that $A' := A + \mathrm{d}f$ satisfies *the Cauchy radiation gauge*.

Sketch of the proof: - We decompose $A = A_0 dt + A_\Sigma$
- A' satisfies the Cauchy radiation gauge if we can solve the system

$$\begin{cases} \mathrm{D}_0 f &= -\delta A \\ \partial_t f|_\Sigma &= -A_0|_\Sigma \\ \Delta_0 f|_\Sigma &= -\delta_\Sigma A_\Sigma|_\Sigma \end{cases}$$

- Hodge-decomposition $A_\Sigma = df' + \delta\beta + h \Rightarrow \exists\, f$ solving $\Delta_0 f = -\delta_\Sigma A_\Sigma$ □

THE GAUGE-FIXED PHASE SPACE

REMARK: f is unique (up to a constant), so the gauge is fixed completely, *i.e.*

$$\frac{\ker(\mathsf{P})}{\mathrm{ran}(\mathsf{K})} \simeq \ker(\mathsf{D}_1) \cap \ker(\mathsf{K}^*) \cap \ker(\mathsf{R})$$

where $\mathsf{R} = U_1 \mathsf{R}_\Sigma \rho_1$ and $\mathsf{R}_\Sigma(a_0, \pi_0, a_\Sigma, \pi_\Sigma) := (a_0, \pi_0, 0, 0)$

PROPOSITION: The following diagram is commutative

$$\mathcal{V}_{\mathrm{P}} := \frac{\ker(\mathsf{K}^*)}{\mathrm{ran}(\mathsf{P})} \xrightarrow{[\mathsf{G}_1]} \frac{\ker(\mathsf{P})}{\mathrm{ran}(\mathsf{K}|_{\Gamma_{\mathsf{R}}})}$$

$$\mathcal{V}_{\mathrm{P}} \xrightarrow{[\mathsf{G}_1]} \frac{\ker(\mathsf{D}_1) \cap \ker(\mathsf{K}^*)}{\mathsf{K}(\ker(\mathsf{D}_0))}, \qquad \mathcal{V}_{\mathrm{P}} \xrightarrow{[\rho_1 \mathsf{G}_1]} \mathcal{V}_\Sigma$$

$$\mathcal{V}_\Sigma := \frac{\ker(\mathsf{K}_\Sigma^\dagger)}{\mathrm{ran}(\mathsf{K}_\Sigma)} \xrightarrow{[U_1]} \frac{\ker(\mathsf{D}_1) \cap \ker(\mathsf{K}^*)}{\mathsf{K}(\ker(\mathsf{D}_0))} \hookrightarrow \frac{\ker(\mathsf{P})}{\mathrm{ran}(\mathsf{K}|_{\Gamma_{\mathsf{R}}})}$$

$$\mathcal{V}_\Sigma \overset{\mathsf{T}_\Sigma}{\dashrightarrow} \mathcal{V}_{\mathrm{R}}$$

$$\mathcal{V}_{\mathrm{R}} := \ker(\mathsf{K}_\Sigma^\dagger) \cap \ker(\mathsf{R}_\Sigma) \xrightarrow{U_1} \ker(\mathsf{D}_1) \cap \ker(\mathsf{K}^*) \cap \ker(\mathsf{R}) \hookrightarrow \frac{\ker(\mathsf{D}_1) \cap \ker(\mathsf{K}^*)}{\mathsf{K}(\ker(\mathsf{D}_0))}$$

We next endow $\mathcal{V}_{\rm R}$ with an Hermitian form $q_{\Sigma,{\rm R}}$

- Decomposing $A = A_0 dt + A_\Sigma$, we set

$$\rho_0 \colon f \mapsto \begin{pmatrix} f|_\Sigma \\ \frac{1}{\mathrm{i}}\partial_t f|_\Sigma \end{pmatrix} \qquad \text{and} \qquad \rho_1 \colon A \mapsto \begin{pmatrix} A_0|_\Sigma \\ \frac{1}{\mathrm{i}}\partial_t A_0|_\Sigma \\ A_\Sigma|_\Sigma \\ \frac{1}{\mathrm{i}}\partial_t A_\Sigma|_\Sigma \end{pmatrix}$$

- By construction $[\rho_1 \mathrm{G}_1] \colon (\mathcal{V}_{\rm P}, q_1) \to (\mathcal{V}_\Sigma, q_{1,\Sigma})$ is an unitary isomorphism

$$q_{1,\Sigma}([\cdot],[\cdot]) = \mathrm{i}([\cdot], \mathrm{G}_{1,\Sigma}[\cdot])_{\mathrm{V}_{\rho_1}} \qquad \mathrm{G}_{1,\Sigma} = \frac{1}{\mathrm{i}} \begin{pmatrix} 0 & -\mathbb{1} & 0 & 0 \\ -\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}$$

- We define $\mathrm{q}_{\Sigma,\mathrm{R}}$ such that $\mathrm{T}_\Sigma : (\mathcal{V}_\Sigma, \mathrm{q}_{1,\Sigma}) \to (\mathcal{V}_{\rm R}, \mathrm{q}_{\Sigma,\mathrm{R}})$ is unitary

$$\mathrm{q}_{\Sigma,\mathrm{R}}(\cdot,\cdot) = \mathrm{i}(\cdot \mathrm{G}_{\Sigma,\mathrm{R}} \cdot)_{\mathrm{V}_{\rho_1}} \qquad \mathrm{G}_{\Sigma,\mathrm{R}} = \frac{1}{\mathrm{i}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}$$

Summing up: unitary isomorphisms $(\mathcal{V}_{\rm P}, q_1) \simeq (\mathcal{V}_\Sigma, q_{1,\Sigma}) \simeq (\mathcal{V}_{\rm R}, q_{\Sigma,\mathrm{R}})$

HOW TO CONTROL THE MICROLOCAL BEHAVIOUR OF T_Σ?

To compute T_Σ we follows this ansatz

$$T_\Sigma = \mathbb{1} - \mathsf{K}_\Sigma(\mathsf{R}_\Sigma \mathsf{K}_\Sigma)^{-1}\mathsf{R}_\Sigma$$

PROPOSITION: Set $\pi_\delta := \mathbb{1} - \mathrm{d}_\Sigma \Delta_0^{-1}\delta_\Sigma$. Then the operator defined by

$$T_\Sigma = \begin{pmatrix} 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{pmatrix} \pi_\delta & 0 \\ 0 & \pi_\delta \end{pmatrix} \end{pmatrix}$$

satisfies the following properties

(i) $T_\Sigma = \mathbb{1} - \mathsf{K}_\Sigma(\mathsf{R}_\Sigma \mathsf{K}_\Sigma)^{-1}\mathsf{R}_\Sigma$ on $\ker(\mathsf{K}_\Sigma^\dagger)$

(ii) $\mathsf{T}_\Sigma^2 = \mathsf{T}_\Sigma$ and $\mathsf{T}_\Sigma|_{\mathcal{V}_\mathrm{R}} = \mathbb{1}$;

(iii) $\ker(\mathsf{T}_\Sigma) = \mathrm{ran}(\mathsf{K}_\Sigma)$;

(iv) $\mathrm{ran}(\mathsf{T}_\Sigma) = \ker(\mathsf{K}_\Sigma^\dagger) \cap \ker(\mathsf{R}_\Sigma)$.

NEW RECIPE FOR CONSTRUCTING HADAMARD STATES

0) By the standard deformation argument, we assume

$$(M, g) \quad \text{to be ultrastatic and of bounded geometry}$$

1) Using pseudodifferential calculus and spectral calculus, we can construct a square root ϵ_i of the Hodge-Laplacian Δ_i satisfying

$$\epsilon_i \pi_\delta = \pi_\delta \varepsilon_i \quad \text{modulo } \Psi^{-\infty}$$

where again $\pi_\delta = \mathbb{1} - \mathrm{d}_\Sigma \Delta_0^{-1} \delta_\Sigma$

2) Finally consider the pseudodifferential projectors $\pi^\pm$ defined by

$$\pi^\pm := \frac{1}{2} \begin{pmatrix} \mathbb{1} & \pm\varepsilon_0^{-1} & 0 & 0 \\ \pm\varepsilon_0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} & \pm\varepsilon_1^{-1} \\ 0 & 0 & \pm\varepsilon_1 & \mathbb{1} \end{pmatrix}$$

THEOREM [M.-Schmid]

The operators $c^{\pm} : \mathcal{V}_\Sigma := \dfrac{\ker K_\Sigma^\dagger}{\operatorname{ran} K_\Sigma} \to L^2(\mathrm{M})$ $\quad c^{\pm} := \mathrm{T}_\Sigma \pi^{\pm} \mathrm{T}_\Sigma$

$$T_\Sigma = \begin{pmatrix} 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{pmatrix} \pi_\delta & 0 \\ 0 & \pi_\delta \end{pmatrix} \end{pmatrix} \qquad \pi^{\pm} := \frac{1}{2} \begin{pmatrix} \mathbb{1} & \pm\varepsilon_0^{-1} & 0 & 0 \\ \pm\varepsilon_0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} & \pm\varepsilon_1^{-1} \\ 0 & 0 & \pm\varepsilon_1 & \mathbb{1} \end{pmatrix}$$

have the following properties:

(i) $(c^{\pm})^\dagger = c^{\pm}$ and $c^{\pm}(\operatorname{ran}(\mathrm{K}_\Sigma)) \subset \operatorname{ran}(\mathrm{K}_\Sigma)$

(ii) $(c^+ + c^-)\mathfrak{f} = \mathfrak{f} \mod \operatorname{ran}(\mathrm{K}_\Sigma) \qquad \forall \mathfrak{f} \in \ker(\mathrm{K}_\Sigma^\dagger)$

(iii) $\pm\, \mathrm{q}_{1,\Sigma}(\mathfrak{f}, c^{\pm}\mathfrak{f}) \geq 0 \qquad \forall \mathfrak{f} \in \ker(\mathrm{K}_\Sigma^\dagger)$

(iv) $\mathrm{WF}'(U_1 c^{\pm}) \subset (\mathcal{N}^{\pm} \cup F) \times \mathrm{T}^*\Sigma$ for $F \subset T^*\mathrm{M}$

In other words,

$$\lambda^{\pm} := \pm \mathrm{i}^{-1} U_1 c^{\pm} \circ (\rho_1 \mathrm{G}_1)$$

are the pseudo-covariances of a quasi-free Hadamard state on $\mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \mathrm{q}_1)$.

Sketch of the proof

(i) Since $\varepsilon_i = \varepsilon_i^*$ are formally self-adjointw.r.t the Hodge-inner product on Σ

$$(\pi^{\pm})^{\dagger} = \mathsf{G}_{1,\Sigma}^{-1}(\pi^{\pm})^{*}\mathsf{G}_{1,\Sigma} = \pi^{\pm}\,,$$

Then $\pi^{\pm}$, T_Σ and also $c^{\pm}$ are formally self-adjoint w.r.t. $\sigma_{1,\Sigma}$.

(ii) $\pi^{+} + \pi^{-} = \mathbb{1}$ on $\Gamma^{\infty}_{\mathsf{H}}(\mathsf{V}_{\rho_1})$ and hence

$$(c^{+} + c^{-})\mathfrak{f} = T_\Sigma^2\mathfrak{f} = T_\Sigma\mathfrak{f} = \mathfrak{f} \quad \mathrm{mod} \quad \mathrm{ran}(\mathsf{K}_\Sigma)$$

for all $\mathfrak{f} \in \ker(\mathsf{K}_\Sigma^{\dagger})$, where in the last step we used that T_Σ is a bijection between $\mathcal{V}_{\mathrm{P}}$ and $\mathcal{V}_\Sigma$ together with $T_\Sigma = \mathbb{1}$ on $\ker\mathsf{R}_\Sigma$.

(iii) we compute

$$\pm\mathrm{q}_{1,\Sigma}(\mathfrak{f}, c^{\pm}f) = \pm\mathrm{q}_{1,\Sigma}(\mathfrak{f}, \mathsf{T}_\Sigma\pi^{\pm}\mathsf{T}_\Sigma f) = \pm\mathrm{q}_{\Sigma,\mathsf{R}}(\mathsf{T}_\Sigma\mathfrak{f}, \pi^{\pm}\mathsf{T}_\Sigma f) \geq 0$$

(iv) follows because $\pi^{\pm}$ commutes with T_Σ modulo a smooth kernel and $\pi^{\pm}$ satisfies the Hadamard condition □

Outlook

WHAT WE HAVE SEEN AND WHAT COMES NEXT?

MAXWELL'S THEORY:

- Gauge fixing is useful for getting positivity and gauge invariance

FUTURE WORK: LINEARIZED GRAVITY

- Gauge fixing completely the linearized gravity on the level of initial data: *Synchronous, de Donder, traceless-gauge, . . .*

- Constructing T_Σ is very challenging from a technical point of view (two-tensors can make life miserable very fast)

- We cannot use the deformation argument, so we need to modify $\pi^{\pm}$ such that the operators $c^{\pm} = T_\Sigma \pi^{\pm} \mathrm{T}_\Sigma$ satisfies the Hadamard conditions

Outlook

WHAT WE HAVE SEEN AND WHAT COMES NEXT?

MAXWELL'S THEORY:

- Gauge fixing is useful for getting positivity and gauge invariance

FUTURE WORK: LINEARIZED GRAVITY

- Gauge fixing completely the linearized gravity on the level of initial data: *Synchronous, de Donder, traceless-gauge, . . .*

- Constructing T_Σ is very challenging from a technical point of view (two-tensors can make life miserable very fast)

- We cannot use the deformation argument, so we need to modify $\pi^\pm$ such that the operators $c^\pm = T_\Sigma \pi^\pm \mathrm{T}_\Sigma$ satisfies the Hadamard conditions

THANKS for your attention!