PARACAUSAL DEFORMATIONS OF LORENTZIAN METRICS AND THEIR CONSEQUENCES IN QUANTUM FIELD THEORY

Simone Murro

Department of Mathematics
University of Paris-Saclay

Séminaire de Géométrie différentielle

Metz, 11th of October 2021

université PARIS-SACLAY | FACULTÉ DES SCIENCES D'ORSAY

DFG Deutsche Forschungsgemeinschaft
German Research Foundation

ACHIEVEMENTS OF THE XX-CENTURY IN PHYSICS

General Relativity: gravitation interaction $\longleftrightarrow$ Lorentzian manifold $(\mathcal{M}, \mathrm{g})$

Quantum Theory: physics of small scale $\longleftrightarrow$ Non-commutative algebras $\mathcal{A}$

$$\downarrow$$

Quantum Field Theory on a Curved Spacetime

- (M, g): Lorentzian manifold
- Ψ: section of vector bundle $E \to M$
- P: (linear) differential operator on E

Quantization: (1) $(\mathrm{Ker}(\mathrm{P}), \sigma) \longrightarrow \mathcal{A}^{\mathrm{obs}}$ (2) $\omega : \mathcal{A}^{\mathrm{obs}} \to \mathbb{C}$

NATURAL QUESTIONS: How much the Physics 'depends' on the metric?

If g and g' are 'related', are the physical theories equivalent?

GOAL of TODAY: Provide a new, interesting notion of relation for Lorentzian metrics

$$g \simeq g' \implies (\mathrm{Ker_g}(\mathrm{P}), \sigma) \simeq (\mathrm{Ker_{g'}}(\mathrm{P'}), \sigma') \implies \mathcal{A}^{\mathrm{obs}}_g \simeq \mathcal{A}^{\mathrm{obs}'}_{g'} \text{ and } \omega_g \text{ “}\simeq\text{” } \omega'_{g'}$$

OUTLINE OF THE TALK

(1) Preliminaries on Lorentzian Geometry

(2) Paracausal deformations of Lorenzian metrics

(3) Møller operators for paracausal related metrics

(4) Conclusion and future outlook

- Joint work with Valter Moretti and Daniele Volpe — (arXiv:2109.06685)

Paracausal deformations of Lorentzian metrics and Møller isomorphisms in algebraic quantum field theory.

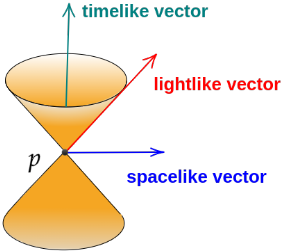
PART (1): Preliminaries in Lorentzian Geometry

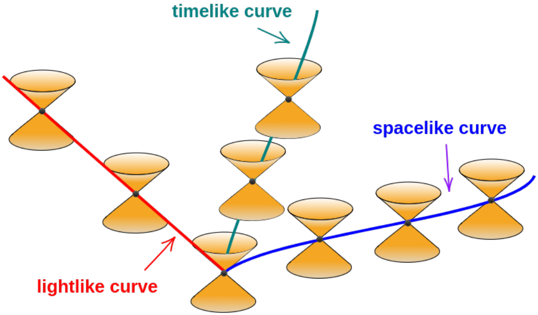
CAUSALITY IN LORENTZIAN GEOMETRY

- (M, g) is a Lorentzian manifold, i.e. $g \in \Gamma(\otimes_s^2 T^*M)$ with signature $(-, +, \dots, +)$
- $\sharp : \Gamma(T^*M) \to \Gamma(TM)$ def by $g(\omega^\sharp, v) = \omega(v) \implies g^\sharp(\cdot, \cdot) := g(\cdot^\sharp, \cdot^\sharp) \in \Gamma(\otimes_s^2 TM)$
- Classification of vectors $v_p \in T_pM$ (and curves $\gamma : I \to M$)

spacelike $g_p(v_p, v_p) > 0$ lightlike $g_p(v_p, v_p) = 0$ timelike $g_p(v_p, v_p) < 0$

open lightcone $V_p^g := \{v_p \text{ timelike}\}$ lightcone $J_g(p) := \{q \in M \,|\, \exists\gamma \text{ causal}\}$





GLOBALLY HYPERBOLIC SPACETIMES

- Spacetime: a connected, time-oriented, smooth Lorentzian $n+1$-manifold (M, g)
- Temporal function: $t \in C^\infty(M, \mathbb{R})$ strictly increasing on future directed causal curve and ∇t is timelike everywhere and past-pointing
- Cauchy hypersurface Σ: if each inextendible timelike curve $\gamma \cap \Sigma = \{pt\}$
- Globally hyperbolic spacetime: M strongly causal and $J^+(p) \cap J^-(q)$ is compact

Theorem [Bernal-Sánchez] (M, g) is globally hyperbolic

$$\Updownarrow$$

$\exists$ a Cauchy temporal function i.e. $t^{-1}(s) := \Sigma_s$ is a Cauchy hypersurface

$$\Updownarrow$$

M isometric to $\mathbb{R} \times \Sigma$ with metric $-\beta^2 dt^2 + h_t$, where $\beta \in C^\infty(\mathrm{M}, (0, \infty))$

Example: Minkoski spacetime $(\mathbb{R}^4, \eta)$, Schwarzchild spacetime $(\mathbb{R}^2 \times \mathbb{S}^2, g_S)$

NOT Example: anti-de Sitter space $(\mathbb{S}^1 \times \mathbb{R}^3, g_{adS})$, Gödel universe $(\mathbb{R}^4, g_G)$

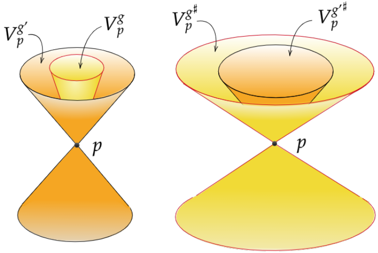
PART (2): Paracausal deformations of Lorentzian metrics

TOWARDS PARACAUSAL DEFORMATION OF LORENTZIAN METRICS

THE 'INCLUSIONS OF OPEN LIGHTCONES' RELATION

$\mathcal{M}_M :=$ {Lorentzian metrics} $\quad \mathcal{T}_M :=$ {time-oriented Lorentzian metrics}

$\mathcal{GH}_M :=$ {globally hyperbolic metrics}



Notation: $g \preceq g'$ if and only if $V_p^g \subset V_p^{g'}$

Few properties for $g \preceq g'$

(1) $g, g' \in \mathcal{M}_M$ and $\lambda, \chi : M \to [0,1]$

- (i) $g \preceq g'$ if and only if $g'^\sharp \preceq g^\sharp$
- (ii) $g_\lambda := (1-\lambda)g + \lambda g'$ and $g_\chi := \left((1-\chi)g^\sharp + \chi g'^\sharp\right)^\flat$ are Lorentzian
- (iii) $g \preceq g_\lambda \preceq g'$ and $g \preceq g_\chi \preceq g'$

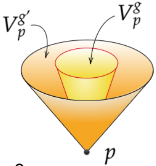
(2) $g \in \mathcal{T}_M$, $g' \in \mathcal{GH}_M$ and $\lambda, \chi : M \to [0,1]$

- (i) g'-Cauchy hypersurfaces are g-Cauchy hypersurfaces
- (ii) $g, g_\lambda, g_\chi \in \mathcal{GH}_M$

SKETCH OF THE PROOF

(1) (ii) - let e_0 be g- and g'-timelike and $\{e_0, e_1, \dots e_n\}$ be a basis of T_pM

- $S :=$span$< e_1, \dots, e_n > \Rightarrow v \in S$ is g'-spacelike $\Rightarrow$ is g-spacelike
- assume $0 \neq \lambda \neq 1$



$$g_p + \frac{\lambda(p)}{1-\lambda(p)} g'_p \equiv G := \begin{bmatrix} h & c^t \\ c & A \end{bmatrix}$$

$$h := g(e_0, e_0) + \frac{\lambda}{1-\lambda} g'(e_0, e_0) < 0 \qquad \text{and} \qquad A := g(v, v) + \frac{\lambda}{1-\lambda} g'(v, v) > 0$$

- Linear algebras shows that $D_4 D_3 D_2 D_1 \begin{bmatrix} h & c^t \\ c & A \end{bmatrix} (D_4 D_3 D_2 D_1)^t = \mathrm{diag}(-1, 1, \dots, 1)$

$$D_1 := \begin{bmatrix} (-h)^{-1/2} & 0^t \\ 0 & I_n \end{bmatrix} \; D_2 := \begin{bmatrix} 1 & 0^t \\ 0 & S \end{bmatrix} \; D_3 := \begin{bmatrix} 1 & 0^t \\ 0 & R \end{bmatrix} \; D_4 := \begin{bmatrix} F & 0 \\ 0 & I_{n-1} \end{bmatrix}$$

for suitable $S \in GL(n, \mathbb{R})$, $R \in O(n, \mathbb{R})$ and $F \in O(2, \mathbb{R})$

(1) (iii) - if v is g-timelike $\Rightarrow$ g'-timelike $\Rightarrow$ g_λ-timelike $\Rightarrow g \preceq g_\lambda$

- if v is g_λ-timelike $\Rightarrow$ g'-timelike or g-timelike ($\Rightarrow$ g'-timelike) $\Rightarrow g_\lambda \preceq g'$

(2) (i) - g'-spacelike Σ is g-spacelike

- inextendible g-timelike γ are inextendible g'-timelike $\Rightarrow \Sigma$ is g-Cauchy

PARACAUSAL DEFORMATION OF LORENTZIAN METRICS

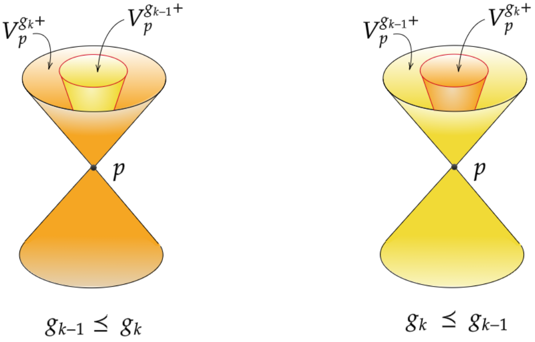
Definition: $\mathcal{GH}_M \ni g, g'$ are paracausal related $g \simeq g'$

if $\exists$ a finite sequence $g, g_1, \ldots, g_n, g' \in \mathcal{GH}_M$ such that

(i) g_k and g_{k+1} are $\preceq$-comparable,

(ii) (a) if $g_k \preceq g_{k+1}$, then $V_p^{g_k +} \subset V_p^{g_{k+1} +}$ for all $p \in$ M,

(b) if $g_{k+1} \preceq g_k$, then $V_p^{g_{k+1} +} \subset V_p^{g_k +}$ for all $p \in$ M.



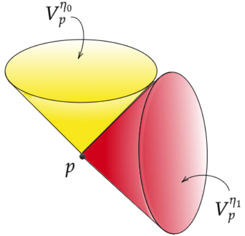
PARACAUSAL DEFORMATIONS OF MINKOWSKI METRICS II

$\mathbb{R}^n$ endowed with the Minkowski metrics

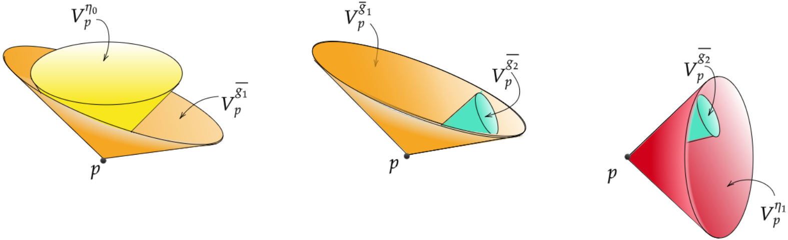
$$\eta_0 = -dt \otimes dt + \sum_{i=1}^{n} dx^i \otimes dx^i$$

$$\eta_1 = -d\tau \otimes d\tau + \sum_{i=1}^{n} dy^i \otimes dy^i$$

where $\tau = x_1$, $y_1 = t$, and $y_k = x_k$ for $k > 1$.



η_0 and η_1 are paracausally related



PARACAUSAL DEFORMATIONS OF MINKOWSKI METRICS II

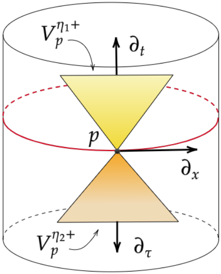
$\mathbb{R} \times \mathbb{S}^1$ endowed with the Minkowski metrics

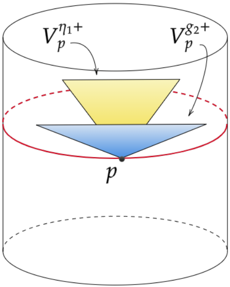
$$\eta_1 = -dt \otimes dt + dx \otimes dx$$

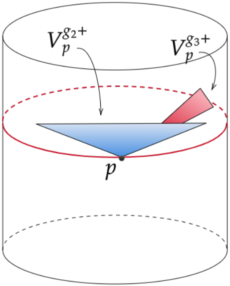
$$\eta_2 = -d\tau \otimes d\tau + dx \otimes dx$$

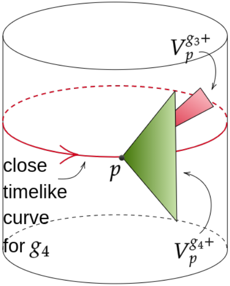
where $\partial_\tau = -\partial_t$

η_1 and η_2 are NOT paracausally related









PROPERTIES OF PARACAUSAL RELATED METRICS I

Proposition: Sufficient conditions for $g \simeq g'$

(I) $V_p^{g+} \cap V_p^{g'+} \neq \emptyset$;

(II) $\exists$ *common* Cauchy temporal function t such that $t^{-1}(s)$ is compact;

(III) $\exists$ g-Cauchy temporal function t s.t.

  (a) $t^{-1}(s)$ are compact and g'-spacelikes,

  (b) dt is g'-past-directed;

Sketch of the Proof (I):

- we have seen that if $g \preceq \hat{g}$ and $\hat{g} \in \mathcal{GH}_M \Rightarrow g \in \mathcal{GH}_M$

-it is enough show exists Lorentzian metric h s.t. $h \preceq g_k$ and $h \preceq g_{k+1}$

step 1: existence smooth vector field X on M such that $X_p \in V_p^{g_k+} \cap V_p^{g_{k+1}+}$

step 2: construction of Lorentzian metric s.t. $X_p \in V_p^{h+} \subset V_p^{g_k+} \cap V_p^{g_{k+1}+}$

$$h_p(v,v) := g_p(v,v') + (a(p)-1)\frac{g(X_p,v)g(X_p,v')}{g(X_p,X_p)}$$

Theorem: $g \simeq g' \iff \exists \{g_i\} \subset \mathcal{GH}_M$ s.t. $V_p^{g_i+} \cap V_p^{g_{k+1}+} \neq \emptyset$ for every $p \in$ M

PROPERTIES OF PARACAUSAL RELATED METRICS II

Proposition: Sufficient conditions for $g \simeq g'$

(I) $V_p^{g+} \cap V_p^{g'+} \neq \emptyset$;

(II) $\exists$ *common* Cauchy temporal function t such that $t^{-1}(s)$ is compact;

(III) $\exists$ g-Cauchy temporal function t s.t.

  (a) $t^{-1}(s)$ are compact and g'-spacelikes,

  (b) dt is g'-past-directed;

Sketch of the Proof (II):

- $g \preceq \hat{g} := \beta_0^{-2} g = -dt \otimes dt + \beta_0^{-2} h_t$ and $g' \preceq \hat{g}' := \beta_1^{-2} g' = -dt \otimes dt + \beta_1^{-2} h'_t$

- since Σ_t is compact $\Rightarrow$ $U\Sigma_t$ is compact

$$f_t(v,x) = \frac{\hat{g}'(v,v)|_x}{\hat{g}(v,v)|_x} = \frac{\beta_1^{-2} h'_t(v,v)|_x}{\beta_0^{-2} h_t(v,v)|_x} \geq C(t) \in (0,1]$$

- $g_C := -dt^2 + C(t)\beta_0^{-2} h_t$ is globally hyperbolic

- $g_C(v,v) \leq g_0(v,v)$ and $g_C(v,v) \leq g_1(v,v)$, $\Rightarrow J^{\pm}_{g_0} \cup J^{\pm}_{g_1} \subset J^{\pm}_{g_C}$.

- Summing up $g \preceq \hat{g} \preceq g_C \succeq \hat{g}' \succeq g'$

PROPERTIES OF PARACAUSAL RELATED METRICS III

Proposition: Sufficient conditions for $g \simeq g'$

(I) $V_p^{g+} \cap V_p^{g'+} \neq \emptyset$;

(II) $\exists$ *common* Cauchy temporal function t such that $t^{-1}(s)$ is compact;

(III) $\exists$ g-Cauchy temporal function t s.t.

- (a) $t^{-1}(s)$ are compact and g'-spacelikes,
- (b) dt is g'-past-directed;

Sketch of the Proof (III):

- $g = -\beta^2 dt \otimes dt + h_t$, $\quad g^\sharp = -\beta^{-2}\partial_t \otimes \partial_t + h_t^\sharp$
- Σ_t is g'-spacelike and $-dt$ is a g'-timelike $\Rightarrow dt \in V_x^{g^{\sharp +}} \cap V_x^{g'^{\sharp +}} \neq \emptyset$
- $g_a^\sharp = -a\beta^{-2}\partial_t \otimes \partial_t + h^\sharp$, s.t. $V_x^{g_a^{\sharp +}} \subset V_x^{g^{\sharp +}}$ and $V_x^{g_a^{\sharp +}} \subset V_x^{g'^{\sharp +}}$
- $g \preceq g_a$ and $g' \preceq g_a$

Theorem: (M, g) and (M, g') are Cauchy-compact. $g \simeq g'$if and only if

$\exists \{g_i\} \subset \mathcal{GH}_M$ s.t. $t_k^{-1}(s)$ is g_{k+1}-spacelikes cpt and dt is g_{k+1}-past-directed

PART (3):
Møller operators for paracausal related metrics

MØLLER OPERATORS FOR PARACAUSAL RELATED METRICS

- E is $\mathbb{K}$-vector bundle over M with an Hermitian fiber metric $\prec \mid \succ$
- $N, N' : \Gamma(E) \to \Gamma(E)$ normally hyperbolic operators associated to $g, g' \in \mathcal{GH}_M$, i.e.

$$\sigma_N(\xi) = -g^\sharp(\xi,\xi) \qquad \sigma_{N'}(\xi) = -g'^\sharp(\xi,\xi)$$

- symplectic form $\sigma^N_{(M,g)} : \ker_{sc}(N) \times \ker_{sc}(N) \to \mathbb{C}$

$$\sigma^N_{(M,g)}(\psi,\phi) = \int_\Sigma \Big(\prec \psi \,|\, \nabla_n \phi \succ - \prec \nabla_n \psi \,|\, \phi \succ \Big) \mathrm{vol}_\Sigma$$

Theorem: If $g \simeq g' \Rightarrow \exists$ isomorphism $R : \Gamma(E) \to \Gamma(E)$, called **Møller operator** s.t.

(a) $R|_{\ker^g_{sc}(N)} : \ker^g_{sc}(N) \xrightarrow{\simeq} \ker^{g'}_{sc}(N')$

(b) $\sigma^{N'}_{g'}(R\psi, R\phi) = \sigma^N_g(\psi,\phi)$ for every $\psi, \phi \in \ker^g_{sc}(N)$

Idea: $g \preceq g' \Longrightarrow N_\chi := (1-\chi)N + \chi N'$ is *norm. hyp.* for $g_\chi := ((1-\chi)g^\sharp + \chi g'^\sharp)^\flat$

$$\left.\begin{aligned} R_+ &:= Id - G^+_{N_\chi}(N_\chi - N) : \ker^g_{sc}(N) \xrightarrow{\simeq} \ker^{g_\chi}_{sc}(N_\chi) \\ R_- &:= Id - G^-_{N'}(N' - N_\chi) : \ker^{g_\chi}_{sc}(N_\chi) \xrightarrow{\simeq} \ker^{g}_{sc}(N') \end{aligned}\right\} \Rightarrow R = R_- R_+$$

QUANTIZATION

(1) assignment of the algebra of observables

$$\ker_{sc}^{g_\chi}(N), \sigma_g^N) \longrightarrow \mathcal{A} := \frac{\bigoplus_n \ker_{sc}^{g}(N)^{\otimes n}}{I_{CCR} = < (\Psi_\psi \otimes \Phi_\phi - \Phi_\phi \otimes \Psi_\psi - \sigma_g^N(\psi, \phi) >}$$

(2) assignment of a state $\omega : \mathcal{A} \to \mathbb{C}$

$$\omega(1_{\mathcal{A}}) = 1 \qquad \text{and} \qquad \omega(\mathfrak{a}^* \mathfrak{a}) \geq 0$$

Remark: Physical states satisfy Hadamard condition $\omega_2 : \mathcal{A}_2 \to \mathbb{C}$

$$WF(\widetilde{\omega}_2) = \{(q, q', p, -p') \in T^*(\mathrm{M} \times \mathrm{M}) \,|\, (q, p) \sim ((q', p')\, p' \triangleright 0\}$$

Theorem:

- If $\mathfrak{g} \simeq g' \Rightarrow \exists$ $*$-isomorphism $\mathcal{R} : \mathcal{A} \to \mathcal{A}'$ s.t. for any $\omega : \mathcal{A} \to \mathbb{C}$

$$\omega' := \omega \circ \mathcal{R} \text{ is Hadamard} \qquad \Longleftrightarrow \qquad \omega \text{ is Hadamard}$$

- for any (M, g) globally hyperbolic, there exists planty of Hadamard states

PART (4): Conclusion and future outlook

TAKE HOME MESSAGE

If $g \simeq g' \Longrightarrow (\mathcal{A}, \omega)$ is 'equivalent' to $(\mathcal{A}', \omega')$

WHAT COMES NEXT?

Conjecture 1

- (M, g) and (M, g') are Cauchy-compact globally hyperbolic ;
- t and t' be Cauchy temporal functions for g and g' ;

$$g \simeq g' \qquad \text{if and only if} \qquad <\partial_t, dt'> > 0 \;\text{ and }\; <\partial_{t'}, dt> > 0$$

Remark: $<\partial_t, dt'> > 0 \Longrightarrow$ integral curve $\gamma = \gamma(t)$ of ∂_t on (M, g') satisfies

$$t'(\gamma(t_2)) > t'(\gamma(t_1)) \qquad \text{for } t_2 > t_1$$

Conjecture 2

- (M, g) and (M, g') are asymptotically flat globally hyperbolic ;
- t and t' be Cauchy temporal functions for g and g' ;

$$<\partial_t, dt'> > 0 \;\text{ and }\; <\partial_{t'}, dt> > 0 \qquad \Longrightarrow \qquad g \simeq g'$$

THANKS FOR YOUR ATTENTION!