A PATHWAY TO NON-COMMUTATIVE GELFAND DUALITY

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Conference

An Invitation to Derived Geometry

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MOTIVATIONS

In the beginning of 20th century

CLASSICAL MECHANICS $\underset{\hbar \to 0}{\overset{\mathcal{Q}_\hbar}{\rightleftarrows}}$ QUANTUM MECHANICS

$f, g \in C^\infty(\mathrm{T}^*M)$ — $\mathfrak{q} = \mathfrak{q}^*,\ \mathfrak{p} = \mathfrak{p}^* \in L(\mathcal{H})$

$[f, g] = 0$ — $[\mathfrak{f}, \mathfrak{g}] = \imath\hbar\,\sigma(\mathfrak{f}, \mathfrak{g})\,\mathbf{1}$

Why not gravity?

CLASSICAL SPACETIME $\underset{\hbar \to 0}{\overset{\mathcal{Q}_\hbar}{\rightleftarrows}}$ QUANTUM SPACETIME

C^∞-Manifolds — ?

Gelfand duality (↕) — noncomm duality (⇣⇡)

Comm. C^∞-Algebra $\underset{\hbar \to 0}{\overset{\mathcal{Q}_\hbar}{\rightleftarrows}}$ Noncomm. C^∞-Algebra

GOAL: A NEW DUALITY FOR NON-COMMUTATIVE RINGS

PLAN OF THE TALK

In this talk, we focus only on the construction of the spectrum:

(I) PROBLEMS IN CONSTRUCTING THE SPECTRUM

(II) DERIVED GEOMETRY: A NEW HOPE

(III) THE SPECTRUM OF A NON-COMMUTATIVE RING

(IV) FUTURE OUTLOOK

Work in progress with
F. Bambozzi (Padova), M. Capoferri (Heriot-Watt), K. Kremnizer (Oxford)
F. Papallo (Genova), M. Vassallo (Genova)

THE FIRST DIFFICULTY: Reyes' no-go theorem

The first step is to define the **spectrum** of a ring. For a commutative complex C^*-algebra $\mathcal{A}$, it is possible to define the Gelfand spectrum

$$\operatorname{Spec}\mathcal{A} := \mathsf{Hom}_{\mathbb{C}\text{-}\mathsf{Alg}}(\mathcal{A}, \mathbb{C}) + \text{weak } *\text{-topology}$$

More generally, for a commutative ring $\mathcal{R}$

$$\operatorname{Spec}\mathcal{R} := \{\text{prime ideals}\} + \text{Zariski topology}\,.$$

It would be natural to extend these definitions to the non-commutative setting

(A) $\mathsf{Rings} \xrightarrow{\operatorname{Spec}^{\mathrm{NC}}} \mathsf{Top}$, $\mathsf{CRings} \to \mathsf{Rings}$, $\mathsf{CRings} \xrightarrow{\operatorname{Spec}} \mathsf{Top}$

(B) $\operatorname{Spec}^{\mathrm{NC}}(\mathcal{A}) = \emptyset$ if and only if $\mathcal{A} = 0$

THEOREM [Reyes]: It does not exists spectrum functor satisfying (A) and (B)

THE SECOND DIFFICULTY: the choice of a good topology

The *Grothendieck topology* is a choice of morphisms on a category $\mathscr{C}$ that makes the objects of $\mathscr{C}$ act like the open sets of a topological space:

DEFINITION: A **Grothendieck topology** is the data of a family of covers s.t.

- if $V \simeq U$, then $\{V \to U\}$ is a cover;
- if $\{U_i \to U\}$ is a cover and $V \to U$ any morphism, then $\{V \times_U U_i \to V\}$ is a cover;
- if $\{U_i \to U\}$ is a cover and for each i, $\{V_{ij} \to U_i\}$ is a cover, then the composition $\{V_{ij} \to U\}$ is a cover.

EXAMPLE: Zariski topology for commutative rings

- *open embeddings:* $A \to B$ flat epimorphism of finite presentation
- *covers:* conservative family of open embeddings $\{A \to B_i\}$ i.e. the product functor $\mathrm{Mod}_A \to \prod_i \mathrm{Mod}_{B_i}$ is conservative

NO-GO : the pushouts of rings is given by the free product of rings, and this operation does not preserve flatness.

DERIVED GEOMETRY: a new hope

KEY FACT: A morphism $A \to B$ in CRings is a *Zariski localization* if and only if it is *homotopical epimorphism*, i.e. $B \otimes_A^{\mathbb{L}} B \simeq B$, of finite presentation

Therefore, we work homotopy category of connective dg-algebras

$$\text{Rings} \hookrightarrow \text{HRings} := \text{Ho}(\text{DGA})^{\leq 0}$$

DEFINITION: We call **formal homotopical Zarisky topology** in HRings

- *open embedding:* $A \to B$ homotopical epimorphism, i.e. $B *_A^{\mathbb{L}} B \simeq B$
- *formal covers:* conservative family of open embeddings $\{A \to B_i\}$ i.e. the product functor $\text{HRings}_A \to \prod_i \text{HRings}_{B_i}$ is conservative

THEOREM: The formal homotopical Zarisky topology is a Grothendieck topology

Is it compatible with classical algebraic geometry? YES!

[Chuang-Lazarev]: for a morphism $A \to B$ in Rings it is equivalent

$$B \otimes_A^{\mathbb{L}} B \simeq B \quad \Longleftrightarrow \quad B *_A^{\mathbb{L}} B \simeq B$$

THE SPECTRUM OF A NON-COMMUTATIVE RING

To $R \in \text{HRings}$ a topological space to $R \in \text{HRings}$, we need to identify open sets, as well intersections and unions.

First attempt: consider a complete join semi-lattice

- $\text{Loc}(R) := \{\text{hom. epi. w. domain } R\}$ • $A \leq B \Leftrightarrow A \to B$ • $A \vee B = A *^{\mathbb{L}}_R B$

↯ Unfortunately, the ideals of $\text{Ouv}(X) = \text{Loc}(R)^{\text{op}}$ do not form always a frame.

Second attempt: consider a posite, namely

$$(\text{Loc}(R), \leq) \text{ endowed with the formal homotopical Zariski topology}$$

✓ Dually, the ideals of $\text{Ouv}(X)$ forms a frame, so can be seen as open sets!
✓ $\text{Ouv}(X)$ + topol. is equivalent to the site of a sober topological space Zar_X.

DEFINITION: For any $R \in \text{HRings}_{\mathbb{Z}}$, we call *non-commutative spectrum* $\text{Spec}^{\text{NC}}(R)$ the topological space equivalent to Zar_X.

THEOREM: The non-commutative spectrum $\text{Spec}^{\text{NC}} : \text{HRings}_{\mathbb{Z}} \to \text{Top}$ is functorial.

EXAMPLES: commutative rings

- if $\mathbb{K}$ is a field, $\mathrm{Spec}^{\mathtt{NC}}(\mathbb{K}) = \star$
- if R is a discrete valuation ring, $\mathrm{Spec}^{\mathtt{NC}}(R) = \mathrm{Spec}_G(R)$
- for the ring of integers $\mathbb{Z}$

$$\mathrm{Loc}(R) \overset{1:1}{\longleftrightarrow} \{\mathbb{Z} \to \mathbb{Z}[S^{-1}], \text{ where } S \text{ is a subset of primes of } \mathbb{Z}\}$$

it turns out that $\mathrm{Zar}_{\mathrm{Spec}(\mathbb{Z})}$ is a distributive lattice, where

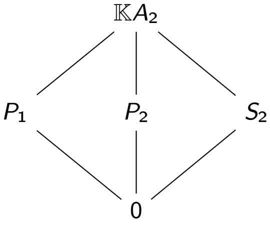
$$\mathrm{Spec}(\mathbb{Z}[S^{-1}]) \wedge \mathrm{Spec}(\mathbb{Z}[T^{-1}]) \cong \mathrm{Spec}(\mathbb{Z}[S^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[T^{-1}]) \cong \mathrm{Spec}(\mathbb{Z}[(S \cup T)^{-1}])$$

$$\mathrm{Spec}(\mathbb{Z}[S^{-1}]) \vee \mathrm{Spec}(\mathbb{Z}[T^{-1}]) \cong \mathrm{Spec}(\mathbb{Z}[(S \cap T)^{-1}]).$$

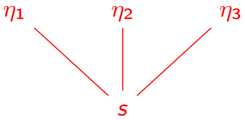
$\mathrm{Spec}^{\mathtt{NC}}(\mathbb{Z}) = \{$the Stone-Cech compactification of $\mathbb{N}$ plus a generic point$\}$

EXAMPLES: non-commutative rings

For the path algebra $R = \mathbb{K}A_2$ over $\mathbb{K}$ of the A_2 quiver, the localization are



the only covers for the homotopy Zariski topology in this case are $\{\mathrm{Id}_{0}\}$, $\{\mathrm{Id}_{P_1}\}$, $\{\mathrm{Id}_{P_2}\}$, $\{\mathrm{Id}_{S_2}\}$, and $\{\mathrm{Id}_{kA_2}\}$ and $\mathrm{Spec}^{\mathtt{NC}}(\mathbb{K}A_2)$ is the spectral space



FUTURE OUTLOOK

To get Gelfand's duality we would like to upgrade the construction of

$$\mathrm{Spec}^{\mathtt{NC}} : \mathsf{HRings} \to \mathsf{Top}$$

to some sort of homotopically ringed space.

The main complication comes from the fact that the base change of algebras and the base change of modules do not agree:
for $A \to B$ localization, $(-) *^{\mathbb{L}}_{A} B$ and $(-) \otimes^{\mathbb{L}}_{A} B$ are not the same.

Therefore, the natural definition of the structure pre-sheaf (i.e. that to a localization $A \to B$ associates B) does not give a sheaf.

But still, there is descent for modules and we can always reconstruct any $M \in \mathsf{HMod}_A$ via the Amistur complex associated with the any cover.

Once this is properly developed we should get Gelfand's duality.

THANKS for your attention!