On the Cauchy problem for symmetric hyperbolic systems on globally hyperbolic manifolds with timelike boundary

Simone Murro

Department of Mathematics
University of Trento

Regensburg working seminars

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Outline of the Talk

- **Preliminaries**
  - **Globally hyperbolic manifolds with timelike boundary**
  - **Symmetric hyperbolic systems**
- **Boundary conditions**
  - **Admissible boundary conditions**
  - **Self-adjoint elliptic boundary conditions**
- **Energy estimates**
- **Existence and uniqueness of smooth solutions**
- **Outlook**

Working in progress

N. Drago, N. Große "Classical Dirac operator with selfadjoint elliptic boundary conditions"

N. Ginoux "Admissible boundary conditions for symmetric hyperbolic systems"

Globally hyperbolic manifolds with timelike boundary

- M is connected, time-oriented, oriented smooth manifold with boundary ∂M
- g and $g|_{\partial M}$ are Lorentzian metric $\Longrightarrow$ M is Lorentzian with timelike ∂M

Few definitions:

- *Temporal function:* $t \in C^\infty(M, \mathbb{R})$ strictly increasing on future directed causal curve and ∇t is timelike everywhere and past-pointing
- *Cauchy hypersurface* Σ: if each inextendible timelike curve $\gamma \cap \Sigma = \{\text{pt}\}$
- *Globally hyperbolic*: M *strongly causal* and $\forall p, q \in M, J^+(p) \cap J^-(q)$ compact

Bernal and Sánchez (2005) – Aké, Flores and Sánchez (2019):

M is globally hyperbolic (with timelike boundary)

$$\Updownarrow$$

Exists a Cauchy temporal function ($t^{-1}(s) := \Sigma_s$ is a Cauchy) and $\nabla t \in T\partial M$

$$\Downarrow$$

M isometric to $\mathbb{R} \times \Sigma$ with metric $-\beta^2 dt^2 + h_t$, where $\beta \in C^\infty(M, (0, \infty))$

Example: Minkoski spacetime $(\mathbb{R}^4, \eta)$, Schwarzchild spacetime $(\mathbb{R}^2 \times \mathbb{S}^2, g_S)$

NOT Example: anti-de Sitter space $(\mathbb{S}^1 \times \mathbb{R}^3, g_{adS})$, Gödel universe $(\mathbb{R}^4, g_G)$

Symmetric hyperbolic systems

- $E \to M$ be a $\mathbb{K}$-vector bundle with finite rank N and sesquilinear fiber metric $\prec \cdot | \cdot \succ$

Definition: a 1^{st} order S is called **symmetric hyperbolic system** if

(S) $\sigma_S(\xi)\colon E_p \to E_p$ is Hermitian with respect to $\prec \cdot | \cdot \succ$, $\forall \xi \in T^*_p M$ and $\forall p \in M$.

(H) $\prec \sigma_S(\tau) \cdot | \cdot \succ$ is positive definite on E_p, for any future-directed timelike $\tau \in T^*_p M$

Furthermore S is **of constant characteristic** if $\dim \ker \sigma_S(\mathtt{n}^\flat)$ is constant, where $\mathtt{n} \perp \partial M$.

Example: $E = \mathbb{C}^N \times \mathbb{R}^n \to (\mathbb{R}^n, \eta)$ with $\prec \, | \, \succ = \langle \, | \, \rangle_{\mathbb{C}^N}$

$$\mathsf{S} := A_0(p)\partial_t + \sum_{j=i}^{n} A_j(p)\partial_{x_j} + B(p)$$

(S) $A_0 = A_0^\dagger\,,\ A_j = A_j^\dagger$ $\qquad$ (H) $\sigma_S(dt + \sum_j \alpha_j dx_j) = A_0 + \sum_{j=1} \alpha_j A_j > 0\,.$

Lemma: If $\prec \cdot | \cdot \succ$ is indefinite and S is a symmetric hyperbolic system, then:

(I) $\langle \cdot | \cdot \rangle := \prec \sigma_S(dt) \cdot | \cdot \succ$ is positive Hermitian metric;

(II) $\mathfrak{S} = -\sigma_S(dt)^{-1}\mathsf{S}$. is symmetric hyperbolic system

(III) Cauchy problem for $\mathfrak{S}$ is equivalent to the Cauchy problem for S.

Example I: The classical Dirac operator

- $M = (M, g)$ is a globally hyperbolic spin manifold with timelike boundary;
- $\mathbb{S}M$ is a *spinor bundle*: $\mathbb{C}$-vector bundle with N-dim. fibers with indefinite sesquilinear metric

$$\prec \cdot \,|\, \cdot \succ : \mathbb{S}_pM \times \mathbb{S}_pM \to \mathbb{C}$$

and a Clifford multiplication, i.e. fiber-preserving map $\gamma\colon TM \to \mathrm{End}(\mathbb{S}M)$

Dirac operator: $\mathrm{D} := \gamma \circ \nabla^{\mathbb{S}} \colon \Gamma(\mathbb{S}M) \to \Gamma(\mathbb{S}M)$ which in local coordinates reads

$$\mathrm{D} = \sum_{\mu=0}^{n} \varepsilon_\mu \gamma(e_\mu) \nabla^{\mathbb{S}}_{e_\mu}$$

- $(e_\mu)_{\mu=0,\dots,n}$ is a local orthonormal Lorentzian frame of TM and $\varepsilon_\mu := g(e_\mu, e_\mu)$
- $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u, v)$ for every $u, v \in T_pM$ and $p \in M$.

Remarks:

(i) Topological obstruction to existence of a spinor bundle;

(ii) Existence of spinor bundles on parallelizable manifolds;

(iii) D is nowhere characteristic.

Example II: The geometric wave operator

-M is a globally hyperbolic with timelike boundary and $g = -\beta^2 dt^2 + h_t$;

- V be an Hermitian vector bundle of finite rank;

- P is a normally hyperbolic operator, i.e. $P = \nabla^*\nabla + c$ and principal symbol σ_P defined by

$$\sigma_P(\xi) = -g(\xi,\xi)\cdot \mathrm{Id}_V\,, \qquad \text{for every } \xi \in T^*M.$$

A norm. hyp. op. **P** can be reduced to $\mathsf{S} : \Gamma(E) \to \Gamma(E)$ defined by

$$\mathsf{S} := (A_0 \nabla_{\partial_t} + A_\Sigma \nabla^\Sigma + C)$$

$$\Psi := \begin{pmatrix} \nabla_{\partial_t} u \\ \nabla^\Sigma u \\ u \end{pmatrix} \quad A_0 := \begin{pmatrix} \frac{1}{\beta^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_\Sigma = \begin{pmatrix} 0 & -\mathrm{tr}_{h_t} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} \text{suitable} \end{pmatrix}.$$

Remarks:

(i) The Cauchy problem for P can be made equivalent to the Cauchy problem for S;

(ii) S is of constant characteristic,

$$\sigma_S(\mathtt{n}^\flat) = \begin{pmatrix} 0 & -\mathtt{n}^\flat \lrcorner & 0 \\ -\mathtt{n}^\flat \otimes & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Admissible boundary conditions

Definition: A boundary space B_{adm} for S is called **admissible** if

- The quadratic form $\Psi \mapsto \prec \sigma_S(\mathtt{n}^\flat)\Psi \,|\, \Psi \succ$ is positive semi-definite on B_{adm};
- rank $B_{adm} = \#$ pointwise non-negative eigenvalues of $\sigma_S(\mathtt{n}^\flat)$ counting multiplicity.

The adjoint boundary space is defined by $B^\dagger_{adm} := \Big(\sigma_{\mathbf{S}}(\mathtt{n}^\flat)(B_{adm})\Big)^\perp$, i.e.

$$\{\Phi \in \Gamma(E|_{\partial M}) \,|\, \text{for any } \Psi \in B_{adm} \text{ it holds } \prec \sigma_{\mathbf{S}}(\mathtt{n}^\flat)\Psi \,|\, \Phi \succ = 0\}\,.$$

Examples for classical Dirac operators:

Lorentzian MIT boundary space is the range of $\pi_{\mathrm{Lor}} := \frac{1}{2}\left(\mathrm{Id} \pm \imath\gamma(\mathtt{n})\right)$

$$\prec \sigma_{\mathbf{D}}(\mathtt{n}^\flat)\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ = \prec \gamma(\mathtt{n})\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ = \imath \prec \pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ$$

Riemannian MIT boundary space is the range of $\pi_{\mathrm{Riem}} := \frac{1}{2}\left(\mathrm{Id} - \frac{1}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\right)$

$$\prec \sigma_{\mathbf{D}}(\mathtt{n}^\flat)\pi_{\mathrm{Riem}}\psi \,|\, \pi_{\mathrm{Riem}}\psi \succ = \prec \gamma(\mathtt{n})\pi_{\mathrm{Riem}}\psi \,|\, \pi_{\mathrm{Riem}}\psi \succ = \frac{1}{\beta} \prec \gamma(\partial_t)\pi_{\mathrm{Riem}}\psi \,|\, \pi_{\mathrm{Riem}}\psi \succ \geq 0$$

Admissible boundary conditions

Definition: A boundary space B_{adm} for S is called **admissible** if

- The quadratic form $\Psi \mapsto \prec \sigma_S(\mathtt{n}^\flat)\Psi \mid \Psi \succ$ is positive semi-definite on B;
- rank $\mathsf{B}_{adm} = \#$ pointwise non-negative eigenvalues of $\sigma_S(\mathtt{n}^\flat)$ counting multiplicity.

The adjoint boundary space is defined by $\mathsf{B}^\dagger_{adm} := \Big(\sigma_{\mathbf{S}}(\mathtt{n}^\flat)(\mathsf{B}_{adm})\Big)^\perp$, i.e.

$$\{\Phi \in \Gamma(E|_{\partial M}) \mid \text{for any } \Psi \in \mathsf{B}_{adm} \text{ it holds } \prec \sigma_{\mathbf{S}}(\mathtt{n}^\flat)\Psi \mid \Phi \succ = 0\}\,.$$

Examples for geometric wave operator:

Neumann like-boundary condition: $\nabla^\Sigma_{\mathtt{n}} u = 0 \quad \Longrightarrow \quad \mathsf{B}^{N.}_{adm} = \ker \begin{pmatrix} 0 & \mathtt{n}\lrcorner & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Transparent boundary condition: $\nabla^\Sigma_{\mathtt{n}} u = -b\nabla_{\partial_t} u \quad \Longrightarrow \quad \mathsf{B}^{T.} = \ker \begin{pmatrix} b & \mathtt{n}\lrcorner & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

NOT example for geometric wave operator:

Robin boundary condition: $\nabla^\Sigma_{\mathtt{n}} u = bu \quad (b \neq 0) \quad \Rightarrow \quad \mathsf{B}^{R.}_{adm} = \ker \begin{pmatrix} 0 & \mathtt{n}\lrcorner & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(the quadratic form is not positive semi-definite)

Self-adjoint elliptic boundary conditions

- M is a globally hyperbolic spin with timelike boundary and $\partial\Sigma$ is compact;
- Dirac operator on spinor bundle $\mathbb{S}M$ reads as $\mathrm{D} = -\gamma(dt)\nabla_{\partial_t} + \mathrm{D}_\Sigma$
- Dirac Hamiltonian $\mathrm{H} := \imath\gamma(dt)^{-1}\mathrm{D}_\Sigma$ $\qquad$ ($\mathrm{D}\Psi = 0 \iff \imath\nabla_{\partial_t}\Psi = \mathrm{H}\Psi$)
- adapted operator $A : \Gamma(\mathbb{S}M|_{\partial M}) \to \Gamma(\mathbb{S}M|_{\partial M})$ formally self-adjoint s. t. $\forall p \in \partial\Sigma_t$ $\forall \xi \in T^*_p\partial\Sigma$

$$\sigma_A(\xi) := \sigma_{\mathbf{H}(t)}(\nu_t^\flat)^{-1} \circ \sigma_{\mathbf{H}(t)}(\xi) \qquad \sigma_{\mathbf{H}(t)}(\nu_t^\flat) \circ A = -A \circ \sigma_{\mathbf{H}(t)}(\nu_t^\flat)\,,$$

- Sobolev-like spaces $\qquad \check{\mathcal{H}}(A(t) := \mathcal{H}^{\frac{1}{2}}_{(-\infty,a)}(A(t)) \oplus \mathcal{H}^{-\frac{1}{2}}_{[a,\infty)}(A(t))$

$$\mathcal{H}^s_I(A(t)) := \left\{ \sum_j \alpha_j\varphi_j(t) \in L^2(\mathbb{S}M|_{\partial\Sigma_t}) \;\middle|\; \sum_{j|\lambda_j\in I} |\alpha_j|^2(1+\lambda_j^2(t))^s < +\infty \right\}, \quad s \in \mathbb{R}\,,$$

Definition: A boundary space B_{ell} for D is called **self-adjoint elliptic** if

(B) $\mathrm{B}_{ell}(t) \subset \check{\mathcal{H}}(A(t))$ is closed;

(E) $\Psi \in \mathcal{H}^k_{loc}(\mathbb{S}M|_{\Sigma_t}) \iff \mathrm{H}_{ell}(t)\Psi \in \mathcal{H}^{k+1}_{loc}(\mathbb{S}M|_{\Sigma_t})\,,$

(SA) $\mathrm{B}_{ell}(t) = \mathrm{B}^\dagger_{ell}(t) := \{\Phi \in \check{\mathcal{H}}(A(t) \,|\, \text{for any } \Psi \in \mathrm{B}_{ell} \text{ it holds } \prec \sigma_{\mathbf{H}(t)}(\mathrm{n}^\flat)\Psi \,|\, \Phi \succ = 0\}$

Examples:

APS boundary conditions $\mathrm{B}_{APS} := \mathcal{H}^{\frac{1}{2}}_{(-\infty,0)}(A(t))$ $\quad$ (self-adjoint $\iff$ $\ker A(t) = \{0\}$).

TAKE HOME MESSAGES

Theorem [Ginoux-M.]

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary

There exists a **unique strong solution** to the Cauchy problem for a symmetric hyperbolic system

$$\begin{cases} \mathsf{S}\Psi = \mathfrak{f} \\ \Psi|_{E|_{\Sigma_0}} = \mathfrak{h} \\ \Psi \in \mathsf{B}_{adm} \end{cases}$$

If $(\mathfrak{f}, \mathfrak{h}) \in \Gamma(E) \times \Gamma(E|_{\Sigma_0})$ satisfy compatibility conditions

(i) S is nowhere characteristic $\Longrightarrow$ the solution is **smooth**;

(ii) S is of constant characteristic $\Longrightarrow$ the solution is **smooth in tangential direction**.

Theorem [Drago-Große-M.]

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary with $\partial\Sigma$ compact

There exists a **unique smooth solution** Ψ to the Cauchy problem for the Dirac operator

$$\begin{cases} \mathsf{D}\Psi = \mathfrak{f} \\ \Psi|_{\mathbb{S}M|_{\Sigma_0}} = \mathfrak{h} \\ \Psi \in \mathsf{B}_{ell} \end{cases}$$

where $(\mathfrak{f}, \mathfrak{h}) \in \check{\Gamma}_c(\mathbb{S}M) \times \check{\Gamma}_c(\mathbb{S}M|_{\Sigma_0}) \subset \Gamma(\mathbb{S}M) \times \Gamma(\mathbb{S}M|_{\Sigma_0})$.

Energy estimates

Theorem (Energy estimates)

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary (and $\partial\Sigma$ compact for elliptic B.C.)
- S is symmetric hyperbolic with B_{adm} while D is Dirac operator with B_{ell}

Then for each $t_0 \in t(M)$ there exists constants $C > 0$ such that for all $t_1 \geq t_0$ it holds

$$\int_{\Sigma_{t_1}} |\Psi|^2 d\mu_{t_1} \leq C e^{C(t_1 - t_0)} \int_{t_0}^{t_1} \int_{\Sigma_s} |\mathrm{D}\Psi|^2 d\mu_s ds + e^{C(t_1 - t_0)} \int_{\Sigma_{t_0}} |\Psi|^2 d\mu_{t_0} \qquad \forall \Psi \in \mathrm{B}_{ell}$$

$$\int_{\Sigma^p_{t_1}} |\Psi|^2 d\mu_{t_1} \leq C e^{C(t_1 - t_0)} \int_{t_0}^{t_1} \int_{\Sigma^p_s} |\mathrm{S}\Psi|^2 d\mu_s ds + e^{C(t_1 - t_0)} \int_{\Sigma_{t^p_0}} |\Psi|^2 d\mu_{t_0} \qquad \forall \Psi \in \mathrm{B}_{adm}$$

where $t \colon M \to \mathbb{R}$ be a Cauchy temporal function and $\Sigma^p_s := J^-(p) \cap \Sigma_s$

Corollary (uniqueness)

If there exists a solution to the Cauchy problem with admissible/elliptic B.C., then it is **unique**

Proof: If $\mathrm{S}(\Psi - \Phi) = 0$, $\Psi, \Phi \in \mathrm{B}_{adm/ell}$ and $\Psi|_{\Sigma_0} = \Phi|_{\Sigma_0} \xrightarrow{\text{Energy estimates}} \Psi - \Phi = 0$

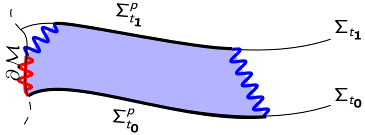
Corollary (finite speed of propagation with B_{adm})

$$\text{supp } \Psi \subset \mathcal{V} := J(\text{supp } f) \cup J(\text{supp } h)$$

Proof: If $p \notin \mathcal{V}$ then $\mathfrak{f}|_{M\setminus\mathcal{V}} = 0$ and $\mathfrak{h}|_{M\setminus\mathcal{V}} = 0 \xrightarrow{\text{Energy estimates}} \Psi|_{M\setminus\mathcal{V}} = 0$

Energy estimates (admissible boundary conditions)

Sketch of the proof:



-n-differential form:

$$\omega := \sum_{j=0}^{n} \Re e \left(\prec \sigma_{\mathbf{S}}(b_j^\flat)\Psi \mid \Psi \succ \right) b_j \lrcorner \mathrm{vol}_M$$

- Stokes' theorem for manifold with Lipschitz boundary yields

$$\int_K d\omega = \int_{\partial K} \omega = \int_{\Sigma^p_{t_1}} \omega - \int_{\Sigma^p_{t_0}} \omega + \int_{\text{red}} \omega + \int_{\text{blue}} \omega$$

- Hyperbolicity of S $\Longrightarrow \int_{\text{blue}} \omega \geq 0$ while $\Psi \in \mathrm{B}_{adm} \Longrightarrow \int_{\text{red}} \omega \geq 0$

$$\int_{\Sigma^p_{t_1}} |\Psi|^2 d\mu_1 - \int_{\Sigma^p_{t_0}} |\Psi|^2 d\mu_0 \leq \int_K d\omega \leq C \int_{t_0}^{t_1} \int_{\Sigma^p_s} (|\Psi|^2 + |\mathrm{S}\Psi|^2) d\mu_s ds$$

- By setting $h(s) := \int_{\Sigma^p_s} |\Psi|^2 d\mu_s \,, \quad \alpha(t_1) := C \int_{t_0}^{t_1} \int_{\Sigma^p_s} (|\Psi|^2 + |\mathrm{S}\Psi|^2) d\mu_s ds + \int_{\Sigma^p_0} |\Psi|^2 d\mu_0$

and using Grönwall, we obtain: $h(t_1) \leq \alpha(t_1) + C \int_{t_0}^{t_1} h(s) ds \leq \alpha(t_1) e^{C(t_1 - t_0)}$

□

Weak and Strong Solutions in a time strip $M_T := t^{-1}(t_1, t_0)$

Definition: We call $\Psi \in \mathscr{H} := \overline{(\Gamma_c(E|_{M_T}), (.\,|\,.)_{M_T})}^{(.\,|\,.)_{M_T}}$

(W) **Weak Solution** if it holds $(\Phi\,|\,\mathfrak{f})_{M_T} = (\mathsf{S}^\dagger\Phi\,|\,\Psi)_{M_T}$

for any $\Phi \in \Gamma_c(E|_{M_T})$ such that $\Phi \in \mathsf{B}^\dagger_{adm/ell}$ and $\Phi|_{\Sigma_{t_1}} = 0 = \Phi|_{\Sigma_{t_0}}$

(S) **Strong Solution** if $\exists\,\{\Psi_k\}_k$, $\Psi_k \in \Gamma(E|_{M_T})$ s.t. $\Psi_k \in \mathsf{B}_{adm/ell}$ on ∂M and

$$\|\Psi_k - \Psi\|_{L^2(M_T)} \xrightarrow{k\to\infty} 0 \quad \text{and} \quad \|\mathsf{S}\Psi_k - \mathfrak{f}\|_{L^2(M_T)} \xrightarrow{k\to\infty} 0$$

Theorem (weak=strong with admissible boundary conditions B_{adm})

- $M = \mathbb{R}\times\Sigma$ globally hyperbolic with timelike boundary
- S is symmetric hyperbolic with admissible boundary conditions B_{adm}

Any weak solution is a strong solution. Moreover, if $(\mathfrak{f}, \mathfrak{h})$ satisfy compatibility conditions

(i) S is nowhere characteristic $\Longrightarrow$ the solution is **smooth**;

(ii) S is of constant characteristic $\Longrightarrow$ the solution is **smooth in tangential direction**;

Comments on the Proof

- Admissible boundary conditions are local, so we can localise
- In Fermi coordinates, we can use the local theory [Phillips-Lax,Rauch,Massey-Rauch].

Existence of a Weak Solution

Theorem (existence weak solutions)

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary (and $\partial\Sigma$ compact for elliptic B.C.)
- S is symmetric hyperbolic with B_{adm} or is Dirac operator with B_{ell}

There exists a weak solution to the Cauchy problem.

Sketch of the proof:

- Energy Estimates: $\|\Phi\|_{L^2(M_T)} \leq c\|\mathsf{S}^\dagger\Phi\|_{L^2(M_T)}$
- The kernel of the operator $\mathsf{S}^\dagger$ acting on $\mathrm{dom}\,\mathsf{S}^\dagger$ is trivial

$$\mathrm{dom}\,\mathsf{S}^\dagger := \{\Phi \in \Gamma_c(E_{M_T}) \mid \Phi|_{\Sigma_{t_1}} = 0, \Phi|_{\Sigma_{t_0}} = 0, \Phi \in \mathsf{B}_{adm/ell}\}$$

- $\ell \colon \mathsf{S}^\dagger(\mathrm{dom}\,\mathsf{S}^\dagger) \to \mathbb{C}$ given by $\ell(\Theta) = (\Phi \mid \mathfrak{f})_{M_T}$ where Φ satisfies $\mathsf{S}^\dagger\Phi = \Theta$
- Energy Estimates $\Rightarrow$ ℓ is bounded:

$$\ell(\Theta) = (\Phi \mid \mathfrak{f})_{M_T} \leq \|\mathfrak{f}\|_{L^2(M_T)} \|\Phi\|_{L^2(M_T)} \qquad \textbf{(Cauchy-Schwarz inequality)}$$

$$\leq \lambda^{-1}\|\mathfrak{f}\|_{L^2(M_T)}\|\mathsf{S}^\dagger\Phi\|_{L^2(M_T)} = \lambda^{-1}\|\mathfrak{f}\|_{L^2(M_T)}\|\Theta\|_{L^2(M_T)},$$

-Hence $(\Phi \mid \mathfrak{f})_{M_T} = \ell(\Theta) \overset{Riesz}{\underset{Thm.}{=}} (\Theta \mid \Psi)_{M_T} = (\mathsf{S}^\dagger\Phi \mid \Psi)_{M_T}$ for all $\Phi \in \mathrm{dom}\mathsf{S}^\dagger$

□

Existence of smooth solutions with elliptic boundary conditions

Theorem (existence smooth solutions with elliptic boundary conditions B_{ell})

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary and $\partial\Sigma$ compact
- D is a Dirac operator with self-adjoint elliptic boundary condition B_{ell}

There exists a **unique smooth solutions** to the Cauchy problem.

Sketch of the proof

- By finite speed of propagation in the bulk, Σ can be chosen compact
- Sobolev-like spaces $\mathcal{H}^\infty_{\rm bc}(E|_{\Sigma_t}) := \bigcap_k \left(\mathcal{H}^k_{\rm bc}(E|_{\Sigma_t}) := \mathrm{dom}\langle \mathsf{H}_{ell}(t)^2 + 1\rangle^{\frac{k}{2}}\right) \subset \Gamma(E)$
- Mollifier $J_\varepsilon(t) := \exp[-\varepsilon\langle \mathsf{H}_{ell}(t)^2 + 1\rangle] \colon \mathcal{H}^\infty_{\rm bc}(E|_{\Sigma_t}) \to \mathcal{H}^\infty_{\rm bc}(E|_{\Sigma_t})$
- Existence of unique solution of regularized equation $\imath\nabla_{\partial_t}\Psi^{(\varepsilon)} = J_\varepsilon(t)\mathsf{H}_{ell}(t)J_\varepsilon(t)\Psi^{(\varepsilon)} + f$,
- Estimates ($\Rightarrow$ equicontinuity) + Ascoli-Arzela ($\Rightarrow$ relatively compactness) + diag. subseq. arg.

$$\{\Psi^{(\varepsilon)}\}_{\varepsilon>0} \supset \{\Psi^{(\varepsilon_j)}\}_{j\in\mathbb{N}} \xrightarrow{j\to+\infty} \Psi \in C^1(\mathbb{R}, \mathcal{H}^\infty_{\rm bc}(E|_{\Sigma_t})) \text{ and satisfies } \imath\nabla_{\partial_t}\Psi = \mathsf{H}_{ell}\Psi + f$$

- By showing that also $\nabla_{\partial_t}\Psi \in C^0(\mathbb{R}, \mathcal{H}^\infty_{\rm bc}(E|_{\Sigma_t}))$ and iterating, we can conclude.

□

Outlook

WHAT WE HAVE SEEN AND WHAT COMES NEXT?

well-posedness of the Cauchy problem for

✓ Dirac operator with self-adjoint elliptic b. c. (Nicolò Drago & Nadine Große)

e.g. classical Dirac operator with APS boundary condition

✓ Symmetric hyperbolic systems with admissible boundary condition (Nicolas Ginoux)

e.g. Wave equation with Neumann and transparent boundary condition

e.g. Classical Dirac operator with MIT boundary conditions

What comes next?

? De Rham-d'Alembert operator (with Nicolas Ginoux)

? Dirac type-operator with general elliptic boundary condition

THANKS for your attention!