# THE QUANTIZATION OF MAXWELL THEORY IN THE CAUCHY RADIATION GAUGE

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joint project with Gabriel Schmid, Ph.D. student in Genoa

## MOTIVATION

- Consider a globally hyperbolic 4-manifold  $M = (\mathbb{R} \times \Sigma, g = -\beta^2 dt^2 + h_t)$
- A gauge theory is a quadruple  $(V_0, V_1, P, K)$  consisting of:
  - (I) two Hermitian bundles  $V_0, V_1$  over M;
- (II) a formally self-adjoint differential operator  $P \colon \Gamma(V_1) \to \Gamma(V_1)$ ;
- (III) a linear differential operator  $\mathsf{K}\colon \Gamma(\mathsf{V}_0)\to \Gamma(\mathsf{V}_1)$  with  $\mathsf{K}\neq 0$  such that
  - (i)  $P \circ K = 0$  (gauge transformation)
  - (ii)  $D_1 := P + KK^* \colon \Gamma(V_1) \to \Gamma(V_1)$  is Green hyperbolic;
  - (iii)  $D_0 := K^*K \colon \Gamma(V_0) \to \Gamma(V_0)$  is Green hyperbolic,

Equivalently, we can work at the level of initial data

(I')  $V_{\rho_i}$  are the bundle of initial data for  $D_i$ 

(III') 
$$\mathsf{K}_{\Sigma} := \rho_1 \mathcal{K} \mathcal{U}_0 \quad \mathsf{K}_{\Sigma}^{\dagger} := \rho_0 \mathcal{K}^* \mathcal{U}_1 \quad \mathsf{G}_i = (\rho_i \mathsf{G}_i)^* \mathsf{G}_{i,\Sigma}(\rho_i \mathsf{G}_i)$$

where  $\rho_i$  and  $U_i$  are the Cauchy data and the Cauchy evolution operator and  $(\rho_i G_i)^*$  is the adjoint of  $(\rho_i G_i)$ 

HOW CAN WE QUANTIZE IT?

Step 1: Construct the classical phase space

and assign  $\forall v \in \mathcal{V}_P$  an element of the abstract unital \*-algebra  $\mathrm{CCR}(\mathcal{V}_P, q_1)$ 

generators: 
$$\Phi(v) \quad \Phi^*(v) \quad \mathbb{1}$$
  
CCR relations:  $\begin{bmatrix} \Phi(v), \Phi(w) \end{bmatrix} = \begin{bmatrix} \Phi^*(v), \Phi^*(w) \end{bmatrix} = 0$   
 $\begin{bmatrix} \Phi(v), \Phi^*(w) \end{bmatrix} = q_1(v, w) \mathbb{1}$ 

Step 2: Construct an Hadamard states  $\omega : CCR(\mathcal{V}_P, q_1) \to \mathbb{C}$  defined by

 $\begin{array}{ll} \mbox{covariances:} & \Lambda^+(v,w) := \omega(\Phi(v)\Phi^*(w)) & \Lambda^-(v,w) := \omega(\Phi^*(w)\Phi(v)) \\ \mbox{Hadamard conditions:} & {\sf WF}'(\Lambda^\pm) \subset \mathcal{N}^\pm \times \mathcal{N}^\pm & \mbox{where:} & \mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^- \end{array}$ 

PROPOSITION [Gérard-Wrochna]: let 
$$c^{\pm} : \Gamma(V_{\rho_1}) \to \Gamma(V_{\rho_1})$$
 be  
(i)  $c^{\pm}(\operatorname{ran}(K_{\Sigma})) \subset \operatorname{ran}(K_{\Sigma})$  and  $(c^{\pm})^{\dagger} = c^{\pm}$  (w.r.t.  $q_{1,\Sigma}$ );  
(ii)  $(c^{+} + c^{-})f = f$  mod  $\operatorname{ran}K_{\Sigma} \forall f \in \ker(K_{\Sigma}^{\dagger})$ ;  
(iii)  $q_{1,\Sigma}(f, c^{\pm}f) = i(f, G_{1,\Sigma}c^{\pm}f)v_{\rho_1} \ge 0 \quad \forall f \in \ker(K_{\Sigma}^{\dagger})$ .  
(iv)  $WF'(U_1c^{\pm}) \subset (\mathcal{N}^{\pm} \cup F) \times T^*\Sigma$  for  $F \subset T^*M$   
Then  $\Lambda^{\pm}([s], [t]) := (s, \lambda^{\pm}t)v_1$  where  $\lambda^{\pm} := (\rho_1G_1)^*iG_{1,\Sigma}c^{\pm}(\rho_1G_1)$   
are pseudo-covariances for a quasifree Hadamard state  $\omega : \operatorname{CCR}(\mathcal{V}_{P}, q_1) \to \mathbb{C}$ .

## Difficulties:

- the fiber metric on  $V_{\rho_1}$  may in general be not positive definite  $\implies$  the positivity (iii) is difficult to achieve
- pseudodifferential calculus works nice with the Hadamard condition (iv), but interact badly with gauge invariance (i) and positivity (iii)

## HOW CAN WE CONSTRUCT HADAMARD STATES?

 $\hookrightarrow$  gauge fixing the degrees of freedom

- (I) The Cauchy radiation gauge
- (II) Hodge-decomposition in Sobolev spaces
- (III) The Complete Gauge Fixing and the Phase space
- (IV) Hadamard states in the Cauchy radiation gauge

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(I) 
$$V_0 = \left( \mathsf{M} \times \mathbb{C}, (\cdot, \cdot)_{V_0} \right)$$
 and  $V_1 = \left( \mathsf{T}^* \mathsf{M} \otimes_{\mathbb{R}} \mathbb{C}, (\cdot, \cdot)_{V_1} \right)$  where  
 $(\cdot, \cdot)_{V_1} := \int_{\mathsf{M}} g^{-1}(\overline{\cdot}, \cdot) \operatorname{vol}_g$ 

(II) set  $P =: \delta d$ , K = d and  $K^* = \delta \Longrightarrow D_1 := \delta d + d\delta$  and  $D_0 = \delta d$ Because ker P is invariant under conformal rescaling we can set

$$\mathsf{M} = \mathbb{R} imes \Sigma$$
  $g = -dt^2 + h_t$ .

DEFINITION:  $A = A_0 dt + A_{\Sigma}$  satisfies Cauchy radiation gauge on a  $\Sigma$  if

$$\delta A = 0$$
 (Lorenz gauge) and  $A_0|_{\Sigma} = \partial_t A_0|_{\Sigma} = 0$ 

**REMARK**: On ultrastatic spacetimes, the following gauge are equivalent:

- (i) A satisfies the Cauchy radiation gauge;
- (ii) A satisfies the *temporal gauge*  $A_0 = 0$  and the *Coulomb gauge*  $\delta_{\Sigma} A_{\Sigma} = 0$ ;
- (iii) The fiber metric  $g^{-1}$  reduces to  $h^{-1}$  in the Cauchy radiation gauge

$$g^{-1}(A,A) = -(A_0,A_0) + h^{-1}(A_{\Sigma},A_{\Sigma}) = h^{-1}(A_{\Sigma},A_{\Sigma}) \ge 0$$

#### HOW TO ACHIEVE THE CAUCHY RADIATION GAUGE?

- Decompose  $A = A_0 dt + A_{\Sigma}$ 

- A' = A + df satisfies the Cauchy radiation gauge if we can solve the system

$$\begin{cases} \mathsf{D}_0 f &= -\delta A \\ \nabla_t f|_{\Sigma} &= -A_0|_{\Sigma} \\ \nabla_t^2 f|_{\Sigma} &= -\nabla_t A_0|_{\Sigma} \end{cases} \iff \begin{cases} \mathsf{D}_0 f &= -\delta A \\ \nabla_t f|_{\Sigma} &= -A_0|_{\Sigma} \\ \Delta_0 f|_{\Sigma} &= -\delta_{\Sigma} A_{\Sigma}|_{\Sigma} \end{cases}$$

- if  $\Sigma$  is compact, by the Hodge decomposition  $A_{\Sigma} = \delta lpha + dg + h$ 

$$\begin{cases} \mathsf{D}_0 f &= -\delta A \\ f|_{\Sigma} &= -g|_{\Sigma} \\ \nabla_t f|_{\Sigma} &= -A_0 \end{cases}$$

- Since  $D_0$  is normally hyperbolic  $\implies \exists ! f$  satisfying the Cauchy problem

IF THE MANIFOLD IS NOT COMPACT?

 $\hookrightarrow {\tt Hodge-decomposition \ in \ Sobolev \ space}$ 

-  $(\Sigma, h)$  be complete *d*-dimensional Riemannian manifold and set

$$L^2_k(\Sigma) := \overline{\Omega^k_c(\Sigma;\mathbb{C})}^{\|\cdot\|_2} \qquad \langle\cdot,\cdot\rangle_2 := \int_{\mathsf{M}} h^{-1}_{(k)}(\bar{\cdot},\cdot) \operatorname{vol}_{\Sigma}$$

-  $\Delta_k = \delta d + d\delta$  is symmetric, positive and essentially self-adjoint and we define

$$\mathsf{E}_k := (\mathbb{1} + \overline{\Delta}_k) \colon \mathcal{D}(\overline{\Delta}_k) \to L^2_k(\Sigma)$$

- We can define the Sobolev space of degree  $s \in \mathbb{R}$  to be the Hilbert space

$$\mathsf{H}^{s}_{k}(\Sigma):=\mathcal{D}(\mathsf{E}^{s/2}_{k}), \qquad \langle\cdot,\cdot\rangle_{\mathsf{H}^{s}}:=\langle\mathsf{E}^{s/2}_{k}\cdot,\mathsf{E}^{s/2}_{k}\cdot\rangle_{L^{2}}\,.$$

- Finally set

$$\Omega^k_s(\Sigma) := \Omega^k(\Sigma; \mathbb{C}) \cap \mathsf{H}^s_k(\Sigma) \quad \text{and} \quad \mathsf{H}^\infty_k(\Sigma) := \bigcap_{s \in \mathbb{R}} \mathsf{H}^s_k(\Sigma)$$

**THEOREM** [M.-Schmid]: For complete Riemannian manifolds  $(\Sigma, h)$  it holds

$$\mathsf{H}^{s}_{k}(\Sigma)\cong\operatorname{Har}^{s}_{k}(\Sigma)\oplus\overline{\mathrm{d}\Omega^{k-1}_{\infty}(\Sigma)}\oplus\overline{\delta\Omega^{k+1}_{\infty}(\Sigma)}\,,$$

where  $\operatorname{Har}_k^s(M) := \ker(\overline{d_k}) \cap \ker(\overline{\delta_k})$  and the closures are taken w.r.t.  $\| \cdot \|_{H^s}$ .

#### Sketch of the proof

(1)  $d: \Omega_{\infty}^{k} \to \Omega_{\infty}^{k+1}$  and  $\delta: \Omega_{\infty}^{k+1} \to \Omega_{\infty}^{k}$  are closable and  $H_{\bullet}^{s}$ -adjoint, i.e.

$$(\overline{\delta}^*A, A')_{\mathsf{H}^s_k} = (A, \overline{d}A')_{\mathsf{H}^s_{k+1}}$$

(2) The sequence

$$0 \to \mathcal{D}_{s}(\overline{\mathrm{d}_{0}}) \xrightarrow{\overline{\mathrm{d}_{0}}} \mathcal{D}_{s}(\overline{\mathrm{d}_{1}}) \xrightarrow{\overline{\mathrm{d}_{1}}} \dots \xrightarrow{\overline{\mathrm{d}_{d-1}}} \mathcal{D}_{s}(\overline{\mathrm{d}_{d}}) \xrightarrow{\overline{\mathrm{d}_{d}}} 0$$

is a well-defined co-chain complex and using that  $\overline{d}$  is closed in H<sup>s</sup>,

$$\begin{aligned} \mathsf{H}_{k}^{\mathsf{s}}(\Sigma) &\cong \ker(\overline{\mathrm{d}}_{k})^{\perp} \oplus \ker(\overline{\mathrm{d}}_{k}) \\ &= \ker(\overline{\mathrm{d}}_{k})^{\perp} \oplus \overline{\operatorname{ran}(\overline{\mathrm{d}}_{k-1})} \oplus (\ker(\overline{\mathrm{d}}_{k}) \cap \operatorname{ran}(\overline{\mathrm{d}}_{k-1})^{\perp}) \\ &= \overline{\operatorname{ran}(\delta_{k+1})} \oplus \overline{\operatorname{ran}(\mathrm{d}_{k-1})} \oplus (\ker(\overline{\mathrm{d}}_{k}) \cap \ker(\overline{\delta}_{k})) \,. \end{aligned}$$

where we used  $\ker(P^*)^{\perp} = \overline{\operatorname{ran}(P)}$  and  $\operatorname{ran}(P)^{\perp} = \ker(P^*)$ 

REMARK:

(i) 
$$\Omega^k_s(\Sigma) := \Omega^k \cap \mathsf{H}^s_k(\Sigma) \cong \operatorname{Har}^s_k(\Sigma) \oplus \left(\Omega^k \cap \overline{\mathrm{d}\Omega^{k-1}_\infty(\Sigma)}\right) \oplus \left(\Omega^k \cap \overline{\delta\Omega^{k+1}_\infty(\Sigma)}\right)$$

(ii) If  $(\Sigma, h)$  is of bounded geometry, H<sup>s</sup> coincides with W<sup>s,2</sup>

(iii) Any form  $\alpha \in \Omega^k(\Sigma) \cap d\Omega^{k-1}_{\infty}(\Sigma)$  is exact

**COROLLARY:** For any  $\omega \in \Omega^1_s(\Sigma)$ , the Poisson equation

 $\Delta_0 f = \delta \omega$ 

has a unique solution on the space  $\{f \in C^{\infty}(\Sigma; \mathbb{C}) \mid \mathrm{d}f \in \overline{\mathrm{d}\Omega^{0}_{\infty}(\Sigma)}^{H^{s}}\}$ 

For  $A = A_0 dt + A_{\Sigma}$  we introduce the spaces

$$\Gamma_{s}(\mathsf{V}_{1}) := \left\{ A \in \Gamma(\mathsf{V}_{1}) \middle| \begin{array}{c} A_{0}|_{\Sigma_{t}}, \partial_{t}A_{0}|_{\Sigma_{t}} \in \Omega_{s}^{0}(\Sigma_{t}) \\ A_{\Sigma}|_{\Sigma_{t}}, \partial_{t}A_{\Sigma}|_{\Sigma_{t}} \in \Omega_{s}^{1}(\Sigma_{t}) \end{array} \right\}$$

COROLLARY: For any  $A \in \Gamma_s(V_1)$  there exists  $f \in \Gamma(V_0)$  w.  $df|_{\Sigma_t} \in \overline{\mathrm{d}\Omega^0_{\infty}(\Sigma_t)}^{H^s}$  such that A' = A + df satisfies the Cauchy radiation gauge.

**REMARK:** f is unique (up to a constant), so the gauge is fixed completely, *i.e.* 

$$\frac{\mathsf{ker}(\mathsf{P})}{\mathsf{ran}(\mathsf{K})} \simeq \mathsf{ker}(\mathsf{D}_1) \cap \mathsf{ker}(\mathsf{K}^*) \cap \mathsf{ker}(\mathsf{R})$$

where  $R = U_1 R_{\Sigma} \rho_1$  and  $R_{\Sigma}(a_0, \pi_0, a_{\Sigma}, \pi_{\Sigma}) := (a_0, \pi_0, 0, 0)$ 

THE GAUGE-FIXED PHASE SPACE

PROPOSITION (phase space): The following diagram is commutative

$$\begin{split} \mathcal{V}_{\mathrm{P}} &:= \frac{\ker(\mathsf{K}^*|_{\Gamma_{tc,s}})}{\operatorname{ran}(\mathsf{P}|_{\Gamma_{tc,s}})} \underbrace{\xrightarrow{[\mathsf{G}_1]}}_{\mathsf{G}_1} \xrightarrow{\operatorname{ker}(\mathsf{P}|_{\Gamma_s})} \underbrace{\xrightarrow{\mathsf{ker}(\mathsf{P}|_{\Gamma_s})}}_{\operatorname{ran}(\mathsf{K}) \cap \Gamma_{s,d}} \\ & \downarrow^{[\rho_1\mathsf{G}_1]} \xrightarrow{[\mathcal{G}_1]} \xrightarrow{\mathsf{f}_1} \underbrace{\mathsf{f}_2}_{\mathsf{F}_1} \xrightarrow{[\mathcal{G}_1]} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \underbrace{\mathsf{f}_2}_{\mathsf{F}_1} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \underbrace{\mathsf{f}_2}_{\mathsf{F}_1} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_1} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \underbrace{\mathsf{f}_2}_{\mathsf{F}_1} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_1} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_1} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2}} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2}_{\mathsf{F}_2} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}} \xrightarrow{\mathsf{f}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}_{\mathsf{F}_2}}_{\mathsf{F}_2}_{$$

We conclude the classical theory, by endowing  $\mathcal{V}_{\mathrm{R}}$  with an Hermitian form  $q_{\Sigma,\mathrm{R}}$ 

- Decomposing 
$$A = A_0 dt + A_{\Sigma}$$
, we set

$$\rho_{0} \colon f \mapsto \begin{pmatrix} f|_{\Sigma} \\ \frac{1}{i} \partial_{t} f|_{\Sigma} \end{pmatrix} \quad \text{and} \quad \rho_{1} \colon A \mapsto \begin{pmatrix} A_{0}|_{\Sigma} \\ \frac{1}{i} \partial_{t} A_{0}|_{\Sigma} \\ A_{\Sigma}|_{\Sigma} \\ \frac{1}{i} \partial_{t} A_{\Sigma}|_{\Sigma} \end{pmatrix}$$

- By construction  $[\rho_1\mathsf{G}_1]\colon (\mathcal{V}_\mathrm{P},q_1) \to (\mathcal{V}_\Sigma,q_{1,\Sigma})$  is an unitary isomorphism

$$q_{1,\Sigma}([\cdot],[\cdot]) = i([\cdot], G_{1,\Sigma}[\cdot])_{V_{\rho_1}} \qquad G_{1,\Sigma} = \frac{1}{i} \begin{pmatrix} 0 & -\mathbb{1} & 0 & 0 \\ -\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}$$

- We define  $\mathrm{q}_{\Sigma,\mathsf{R}}$  such that  $\mathsf{T}_\Sigma:(\mathcal{V}_\Sigma,\mathrm{q}_{1,\Sigma})\to(\mathcal{V}_\mathrm{R},\mathrm{q}_{\Sigma,\mathsf{R}})$  is unitary

**Summing up:** unitary isomorphisms  $(\mathcal{V}_{\mathrm{P}}, q_1) \simeq (\mathcal{V}_{\Sigma}, q_{1,\Sigma}) \simeq (\mathcal{V}_{\mathrm{R}}, q_{\Sigma,\mathsf{R}})$ 

HOW TO CONTROL THE MICROLOCAL BEHAVIOUR OF  $T_{\Sigma}$ ?

To compute  $\mathsf{T}_\Sigma$  we follows this ansatz

$$\mathcal{T}_{\Sigma} = \mathbb{1} - \mathsf{K}_{\Sigma}(\mathsf{R}_{\Sigma}\mathsf{K}_{\Sigma})^{-1}\mathsf{R}_{\Sigma}$$

**PROPOSITION:** Let  $(\Sigma, h)$  be a Riemannian manifold and  $\pi_{\delta} := \mathbb{1} - d_{\Sigma} \Delta_0^{-1} \delta_{\Sigma}$ . There exists a map  $\mathsf{T}_{\Sigma} : \mathcal{H}_S \to \mathcal{H}_S$  defined by

$$T_{\Sigma} = egin{pmatrix} 0_{2 imes 2} & 0_{2 imes 2} \ 0_{2 imes 2} & \left( egin{pmatrix} \pi_{\delta} & 0 \ 0 & \pi_{\delta} \end{pmatrix} \end{pmatrix}$$

satisfies the following properties

WE CAN NOW CONSTRUCT HADAMARD STATES

0)By the standard deformation argument, we assume

(M, g) to be ultrastatic and of bounded geometry

1) Replace the phase space  $(\mathcal{V}_{\rm P},q)$  with the space of initial data  $(\mathcal{V}_{\Sigma},q_{\Sigma})$ 

$$\rho G : (\mathcal{V}_{\mathrm{P}}, \mathrm{q}) \xrightarrow{\simeq} (\mathcal{V}_{\Sigma}, \mathrm{q}_{\Sigma}) \qquad \mathrm{q}_{\Sigma}(\cdot, \cdot) := (\cdot, \mathrm{i} \mathsf{G}_{\Sigma} \cdot) \quad \mathsf{G} = (\rho G)^* \mathsf{G}_{\Sigma}(\rho G)$$

2) Construct an 'approximate' square root of the Hodge-Laplacian:

microlocal factorization of  $U_{\Box} = U_{(\partial_t + i\varepsilon)}\pi^+ + U_{(\partial_t - i\varepsilon)}\pi^-$ 

# INTERMEZZO: pseudodifferential calculus I/II

The differential operator  $d/dx: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$  can be written as

$$rac{d}{dx}f(x) = rac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{ikx}k\hat{f}(k)\,dk$$

hence a m-order differential operator A with constant coefficient reads as

$$Pf(x) = rac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} p(x,k) \hat{f}(k) dk \qquad p(x,k) = \sum_{\alpha \leq m} a_{\alpha}(x) k^{\alpha}$$

The Kohn-Nirenberg quantization is the natural generalization

$$S_{1,0}^{m} \ni p(x,k) \mapsto P\left(x,\frac{d}{dx}\right) := Op(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ik(x-y)} p(x,k) f(y) dy \, dk \in \Psi^{m}(\mathbb{R})$$

where the symbol p(x, k) is promoted to a smooth function in the class

$$S_{1,0}^{m} := \left\{ p \in C^{\infty}(\mathbb{R} \times \mathbb{R}) \left| \left| \frac{d^{\alpha}}{dx^{\alpha}} \frac{d^{\beta}}{dk^{\beta}}(p(x,k)) \right| \leq C_{\alpha\beta} \langle k \rangle^{m-|\beta|} \ \forall \alpha, \beta \in \mathbb{N} \right\} \right\}$$

# INTERMEZZO: pseudodifferential calculus II/II

### NICE PROPERTIES:

- The  $\Psi$ DO-calculus transforms covariantly under local diffeomorphisms:
  - $\psi: \mathbb{R}^n \to \mathbb{R}^n$  differomorphism
  - $U_i \subset \mathbb{R}^n$  precompact open sets and  $\chi_i \in \mathcal{C}^\infty_c(\mathbb{R}^n)$  s.t.  $\chi_i|_{U_i} = 1$
  - $\Rightarrow$  For  $A \in \Psi^m(U_1)$  we have  $\chi_1 A \psi^*(\chi_2 u) = B u \in \Psi^m(U_2)$
  - $\Rightarrow$  the definition of  $\Psi DO$  extends on smooth manifolds

• Let 
$$S^{-\infty} := \bigcap_m S^m_{1,0}$$
 and  $\Psi^{-\infty}(M)$  accordingly:  
 $\Rightarrow A : D'(M) \to C^{\infty}(M)$  is smoothing if and only if  $A \in \Psi^{-\infty}(M)$   
 $\Rightarrow WF(Au) = \emptyset$  for any  $u \in D'(M)$ 

- If M compact and  $A \in \Psi^m(M)$  and  $B \in \Psi^n(M)$ 
  - $\Rightarrow A \circ B \in \Psi^{m+n}$
  - $\Rightarrow$  For polyhomogeneous symbols i.e.  $\sigma_P \sim \sum_j \alpha_j k^j \Rightarrow \sigma_{AB} = \sigma_A \circ \sigma_B \in S_{ph}^{m+n}$

The  $\Psi$ DO-calculus can be extended on *manifolds of bounded geometry* 

#### CONSTRUCTION OF AN 'APPROXIMATE' SQUARE ROOT OF THE LAPLACIAN

(sketch of the proof)

- Let  $M = \mathbb{R} \times \Sigma$  with  $\Sigma$  of bounded geometry
- The closure of the Laplacian  $\overline{\Delta}$  with domain  $H^2(\Sigma)$  is self-adjoint on  $L^2(\Sigma)$
- We fix  $\chi \in C^\infty_c(\mathbb{R})$  with  $\chi(0) = 1$  and set  $\chi_R(\lambda) = \chi(R^{-1}\lambda)$  for  $R \ge 1$
- We get  $\chi_R(\overline{\Delta}) \in \Psi^{-\infty}(\Sigma)$  and we set  $r_{-\infty} = R\chi_R(\overline{\Delta})$
- By the spectral calculus we find R>1 s. t.  $\overline{\Delta}+r_{-\infty}$  is *m*-accreative
- By standard results of Kato,  $\overline{\Delta} + r_{-\infty}$  has a unique *m*-accreative square root

$$\varepsilon = \varepsilon^*$$
  $\exists ! \varepsilon^{-1} \in \Psi^{-1}$   $\varepsilon^2 = \Delta + r_{-\infty}$ 

**3**) The square root  $\epsilon_i$  of the Hodge-Laplacian  $\Delta_i$  has to satisfy

 $\epsilon_i \pi_\delta = \pi_\delta \varepsilon_i \mod \Psi^{-\infty}$ 

where again  $\pi_{\delta} = \mathbb{1} - d_{\Sigma} \Delta_0^{-1} \delta_{\Sigma}$ 

4) Finally consider the pseudodifferential projectors  $\pi^{\pm}$  defined by

$$\pi^{\pm} := \frac{1}{2} \begin{pmatrix} 1 & \pm \varepsilon_0^{-1} & 0 & 0 \\ \pm \varepsilon_0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \pm \varepsilon_1^{-1} \\ 0 & 0 & \pm \varepsilon_1 & 1 \end{pmatrix}$$

THEOREM [S.M., Schmid] Consider the operators  $c^{\pm} := \mathsf{T}_{\Sigma} \pi^{\pm} \mathsf{T}_{\Sigma}$ . Then  $\lambda^{\pm} := (\rho_1 \mathsf{G}_1)^* \lambda_{\Sigma}^{\pm}(\rho_1 \mathsf{G}_1) \quad \text{where} \quad \lambda_{\Sigma}^{\pm} := \pm \mathrm{i} \mathsf{G}_{1,\Sigma} c^{\pm}$ 

are the pseudo-covariances of a quasi-free Hadamard state on  ${\rm CCR}({\mathcal V}_{\rm P}, q_1).$ 

### Sketch of the proof

(i) Since  $\varepsilon_i = \varepsilon_i^*$  are formally self-adjoint w.r.t the Hodge-inner product on  $\Sigma$ 

$$(\pi^{\pm})^{\dagger} = \mathsf{G}_{\mathbf{1},\Sigma}^{-1}(\pi^{\pm})^*\mathsf{G}_{\mathbf{1},\Sigma} = \pi^{\pm} \,,$$

Then  $\pi^{\pm}$ ,  $T_{\Sigma}$  and also  $c^{\pm}$  are formally self-adjoint w.r.t.  $\sigma_{1,\Sigma}$ . (ii)  $\pi^{+} + \pi^{-} = 1$  and hence

$$(c^+ + c^-)\mathfrak{f} = T_{\Sigma}^2\mathfrak{f} = T_{\Sigma}\mathfrak{f} = \mathfrak{f} \mod \operatorname{ran}(\mathsf{K}_{\Sigma}|_{\Gamma_{\mathbf{H}}^{\infty}})$$

for all  $\mathfrak{f} \in \text{ker}(\mathsf{K}_{\Sigma}^{\dagger})$ , where in the last step we used that  $\mathsf{T}_{\Sigma}$  is a bijection between  $\mathcal{V}_{\mathrm{P}}$  and  $\mathcal{V}_{\Sigma}$  together with  $\mathcal{T}_{\Sigma} = \mathbb{1}$  on  $\text{kerR}_{\Sigma}$ .

(iii) we compute

$$\pm \mathrm{q}_{\mathbf{1},\Sigma}(\mathfrak{f},c^{\pm}f) = \pm \mathrm{q}_{\mathbf{1},\Sigma}(\mathfrak{f},\mathsf{T}_{\Sigma}\pi^{\pm}\mathsf{T}_{\Sigma}f) = \pm \mathrm{q}_{\Sigma,\mathsf{R}}(\mathsf{T}_{\Sigma}\mathfrak{f},\pi^{\pm}\mathsf{T}_{\Sigma}f) \geq 0$$

(iv) follows because  $\pi^\pm$  commutes with  $\mathsf{T}_\Sigma$  modulo a smooth kernel and  $\pi^\pm$  satisfies the Hadamard condition

## WHAT WE HAVE SEEN AND WHAT COMES NEXT?

## MAXWELL'S THEORY:

- Gauge fixing is useful for getting positivity and gauge invariance, but "the price to pay" is working with smooth, Sobolev initial data

- For generic manifold , we can construct Hadamard projectors  $\pi^{\pm}$ , but it is not clear that they commute with  $T_{\Sigma}$  (even modulo smoothing)

## FUTURE WORK: LINEARIZED GRAVITY

- Gauge fixing completely the linearized gravity on the level of initial data: *Synchronous, de Donder, traceless-gauge, ...* 

- Constructing  $\mathsf{T}_\Sigma$  is very challenging from a technical point of view (two-tensors can make life miserable very fast)

- We cannot use the deformation argument, so we need to modify  $\pi^{\pm}$  such that the operators  $c^{\pm} = T_{\Sigma}\pi^{\pm}T_{\Sigma}$  satisfies the Hadamard conditions

### THANKS for your attention!