# A PATHWAY TO NON-COMMUTATIVE GELFAND DUALITY

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*What happen if we localise with arbitrary accuracy on a classical spacetime?*

- In classical spacetime, Heisenberg's commutation relations
$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2} \qquad \Delta t \Delta E \geq \frac{\hbar}{2}$$
suggests that *localization* transfers energy to the system
- The associated energy–momentum tensor generates a gravitational field
$$\mathrm{Ric} + (\Lambda - \frac{1}{2}\mathrm{scal})\mathrm{g} = \mathrm{T}$$
- High energy density induced by localisation might produce a closed horizon

*Doplicher, Fredenhagen & Roberts* proposed a quantum spacetime subjected to
$$\Delta t \cdot \sum_{i=1}^{3} \Delta x_i \gtrsim \lambda_P^2 \qquad \sum_{i<j=1}^{3} \Delta x_i \Delta_j \gtrsim \lambda_P^2$$

short answer: *non-commutativity!*

# MOTIVATIONS

*How to recover the geometry?*

$$
\begin{tikzcd}
\text{CLASSICAL SPACETIME} \arrow[r, shift left, "\mathcal{Q}_\hbar"] & \text{QUANTUM SPACETIME} \arrow[l, shift left, "\hbar \to 0"] \\
\text{LocCptHaus} \arrow[d, leftrightarrow, "\text{Gelfand}"', "\text{duality}"] & ? \arrow[d, leftrightarrow, dashed, "\text{noncomm}"', "\text{duality}"] \\
\text{Comm. } C^*\text{-Alg}_{\mathbb{C}} \arrow[r, shift left, "\mathcal{Q}_\hbar"] & \text{Noncomm. } C^*\text{-Alg}_{\mathbb{C}} \arrow[l, shift left, "\hbar \to 0"]
\end{tikzcd}
$$

To implement a duality, we need

1. a good notion of *spectrum* $\quad \mathrm{Spec} : C^*\mathsf{Alg}_{\mathbb{C}} \to \mathsf{Top}$
2. a good notion of a *noncommutative sheaf* $\quad \mathsf{Sh} : \mathsf{Ouv} \to C^*\mathsf{Alg}_{\mathbb{C}}$

GOAL of the TALK: DUALITY FOR NON-COMMUTATIVE RINGS

# PLAN OF THE TALK

(I) PROBLEMS IN CONSTRUCTING THE SPECTRUM

(II) DERIVED GEOMETRY: A NEW HOPE

(III) THE SPECTRUM OF A NON-COMMUTATIVE RING

(IV) FUTURE OUTLOOK

**Based on**

*"Noncommutative Gelfand Duality: the algebraic case"* arXiv:2411.11816

j. w. F. Bambozzi (U. Padova) and M. Capoferri (U. Milano/Heriot-Watt U.)

*"Noncommutative Gelfand Duality: the analytic case"* in preparation

j. w. F. Bambozzi (U. Padova), F. Papallo (U. Genova) and S. Rosarin (U. Genova)

# THE GROTHENDIECK SPECTRUM OF A COMMUTATIVE RING

- For <u>commutative</u> ring, Grothendieck introduced a *topology* on $\mathrm{CRings}^{\mathrm{op}}$

DEFINITION: A **Grothendieck topology** is the data of *covers* $\{U_i \to U\}$ s.t.

1. if $V \simeq U$, then $\{V \to U\}$ is a cover;
2. for any $V \to U$ and cover $\{U_i \to U\}_i$ then $\{V \times_U U_i \to V\}_i$ is a cover;
3. if $\{U_i \to U\}_i$ and $\{V_{ij} \to U_i\}_j$ are covers, then $\{V_{ij} \to U\}_{ij}$ is a cover.

- For the prime spectrum of a commutative ring

$$\mathrm{Spec}\,\mathcal{R} := \{\text{prime ideals}\} + \text{Zariski topology}$$

the topology is defined with this choice of morphisms:

- *open localization:* $A \to B$ flat epimorphism of finite presentation
- *covers:* conservative family of open localization $\{A \to B_i\}$
  i.e. the product functor $\mathrm{Mod}_A \to \prod_i \mathrm{Mod}_{B_i}$ is conservative

# DIFFICULTIES IN THE NONCOMMUTATIVE FRAMEWORK

- Condition 2. encodes the functoriality of the construction:

for any $A \to C$ and any cover $\{A \to B_i\}_i$ then $\{C \to C \otimes_A B_i\}$ is a cover for $C$

$\lightning$ in Rings, the *pushout* (free product) does not always preserve flatness $\lightning$

**PROBLEM 1** : *What is a good Grothendieck topology for noncommutative rings?*

- To bypass the problem, we could try to extend the Grothendieck spectrum

$$\begin{tikzcd} \mathsf{Rings} \arrow[r, "\mathrm{Spec}^{\mathrm{ext}}"] & \mathsf{Top} \\ \mathsf{CRings} \arrow[u] \arrow[ur, "\mathrm{Spec}_G"'] & \end{tikzcd}$$

**NO-GO THEOREM [Reyes]**: It does not exists spectrum functor such that

(A) $\mathrm{Spec}^{\mathrm{ext}}(\mathcal{A}) = \mathrm{Spec}_G(\mathcal{A})$ for any $\mathcal{A} \in$ CRings

(B) $\mathrm{Spec}^{\mathrm{ext}}(\mathcal{A}) = \emptyset$ if and only if $\mathcal{A} = 0$

## SMALL DIGRESSION ON THE GELFAND SPECTRUM (work in progress)

- the Gelfand spectrum of a unital commutative $C^*$-algebra is given by

$$\mathrm{Spec}\,\mathcal{A} := \{\text{characters}\} + \text{weak } *\text{-topology}$$

- this time *ring epimorphisms* correspond to *closed embeddings*

$$\varphi : C(X) \to C(Y) \simeq \frac{C(X)}{\mathrm{Ker}\,\varphi} \qquad \Longleftrightarrow \qquad \iota : Y \hookrightarrow X$$

- the set of points are epimorphisms $\varphi_x : C(X) \to \mathbb{C}$

$\mathrm{Ker}\,\varphi_x = \{f \in C(X) | f(x) = 0\}$ is a non-unital $C^*$Algebra $\Leftrightarrow$ open subsets

- the set of epimorphisms forms a complete join semilattice $(\{\varphi_i\}, \leq, \widehat{\otimes}_{C(X)})$

  and dually, we get a complete meet semilattice $(\{\iota_i\}, \subseteq, \cap)$ which is bounded

  $\Longrightarrow$ *coframe of closed sets* $\Longrightarrow$ $\left(\text{comm}\, \mathbb{C}^*\text{-}Alg_{\mathbb{C}}\right)^{\mathrm{op}} \subset \text{Top}$

PROBLEMS for general $C^*$-$Alg_{\mathbb{C}}$ : *1. Epimorphisms are few.*

*2. The lattice might not be distributive.* *3. Reyes' no-go theorem.*

# DERIVED GEOMETRY IN A NUTSHELL

*How 'to relax' the Zariski topology?*

- Instead of working with Rings, we should consider its *homotopy category*:
  - $\mathrm{Rings}_{\mathbb{Z}} = \mathrm{A}lg(\mathrm{Ab}, \otimes_{\mathbb{Z}})$ and $\mathrm{M}od_{\mathbb{Z}}$ category of modules over $\mathrm{Rings}_{\mathbb{Z}}$
  - $\mathrm{C}h^{\leq 0}(\mathrm{M}od_{\mathbb{Z}})$ category of connective complex chain over $\mathrm{M}od_{\mathbb{Z}}$
  - $DGA^{\leq 0}_{\mathbb{Z}} = (\mathrm{C}h^{\leq 0}(\mathrm{M}od_{\mathbb{Z}}), \otimes_{\mathbb{Z}})$ category of dg-algebras, i.e.

$$A^n \otimes_{\mathbb{Z}} A^m \to A^{n+m} \qquad d(a \otimes b) = d(a) \otimes b + (-1)^n \otimes b$$

  - lastly we identify dg-algebras with the same homology

$$\mathrm{HRings}_{\mathbb{Z}} := DGA^{\leq 0}_{\mathbb{Z}}[W^{-1}]$$

- fully faithful inclusion Rings $\hookrightarrow$ HRings given by

$$R \mapsto R^{\bullet} := \Big(0 \to R \to 0\Big)$$

- $\mathrm{HRings}_{\mathbb{Z}}$ has *homotopy pushout* computed using *projective resolutions*

$$A^{\bullet} *^{\mathbb{L}}_{\mathbb{Z}} B^{\bullet} \simeq (P \to A)^{\bullet} *_{\mathbb{Z}} (P \to B)^{\bullet}$$

# FORMAL HOMOTOPICAL ZARISKI TOPOLOGY

[Toen-Vezzosi]: A morphism $A \to B$ in CRings is a *Zariski localization* if and only if it is *homotopical epimorphism*, i.e. $B \otimes_A^{\mathbb{L}} B \simeq B$, of finite presentation

**DEFINITION:** We call **formal homotopical Zarisky topology** in HRings

- *open localization:* $A \to B$ homotopical epimorphism, i.e. $B *_A^{\mathbb{L}} B \simeq B$
- *covers:* conservative (finite) family of open localizations $\{A \to B_i\}$ i.e. the product functor $\mathsf{HRings}_A \to \prod_i \mathsf{HRings}_{B_i}$ is conservative

**THEOREM**: the **form. hom. Zar. top.** is a Grothendieck topology on $\mathsf{HRings}^{\mathrm{op}}$

Is it (mostly) compatible with classical algebraic geometry? **YES!**

[Chuang-Lazarev]: for a morphism $A \to B$ in HRings it is equivalent

$$B \otimes_A^{\mathbb{L}} B \simeq B \quad \Longleftrightarrow \quad B *_A^{\mathbb{L}} B \simeq B$$

## THE SPECTRUM OF A NON-COMMUTATIVE RING

The lattice of *homotopical Zariski localizations* is not distributive!

*How to construct a topological space?*

- Consider the formal duality functor $\mathrm{Spec} : \mathsf{HRings} \to \mathsf{HRings}^{\mathrm{op}}$

$$A \mapsto B_i \qquad \text{and set } Y_i := \mathrm{Spec}(B_i) \;\; X := \mathrm{Spec}(A)$$

- The localizations define the bounded meet semilattice $(\mathsf{Ouv}(X), \cap)$

$$\bullet\, \mathsf{Ouv}(X) := \mathsf{Loc}^{\mathrm{op}} \qquad \bullet\;\; Y_i \cap Y_j = Spec(B_i *_A^{\mathbb{L}} B_j)$$

- Now we follow Jonhstone and 'transform covers to joins':

$$\mathfrak{J} : (\mathsf{Ouv}(X), \cap) \to \mathrm{Frm} \qquad \begin{cases} \text{(i) if } s \leq t \text{ and } t \in \mathfrak{J} \Rightarrow s \in \mathfrak{J}; \\ \text{(ii) if } \{s_i \to s\} \text{ and } s_i \in \mathfrak{J} \Rightarrow s \in \mathfrak{J} \end{cases}$$

**DEFINITION:** $\mathrm{Spec}^{\mathrm{NC}}(R)$ is the topological space given by the frame of ideals.

**THEOREM:** The non-commutative spectrum $\mathrm{Spec}^{\mathrm{NC}} : \mathsf{HRings}_{\mathbb{Z}} \to \mathsf{Top}$ is functorial.

# COMMUTATIVE EXAMPLES I

- if $\mathbb{K}$ is a field, $\mathrm{Spec}^{\mathtt{NC}}(\mathbb{K}) = \star$
- if $R$ is a discrete valuation ring, $\mathrm{Spec}^{\mathtt{NC}}(R) = \mathrm{Spec}_G(R)$
- for the ring of integers $\mathbb{Z}$

$$\mathrm{Loc}(R) \overset{1:1}{\longleftrightarrow} \{\mathbb{Z} \to \mathbb{Z}[S^{-1}], \text{ where } S \text{ is a subset of primes of } \mathbb{Z}\}$$

it turns out that $\mathrm{Zar}_{\mathrm{Spec}(\mathbb{Z})}$ is a distributive lattice, where

$$\mathrm{Spec}(\mathbb{Z}[S^{-1}]) \wedge \mathrm{Spec}(\mathbb{Z}[T^{-1}]) \cong \mathrm{Spec}(\mathbb{Z}[S^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[T^{-1}]) \cong \mathrm{Spec}(\mathbb{Z}[(S \cup T)^{-1}])$$

$$\mathrm{Spec}(\mathbb{Z}[S^{-1}]) \vee \mathrm{Spec}(\mathbb{Z}[T^{-1}]) \cong \mathrm{Spec}(\mathbb{Z}[(S \cap T)^{-1}]).$$

$$\mathrm{Spec}^{\mathtt{NC}}(\mathbb{Z}) = \{\text{the Stone-Cech compactification of } \mathbb{N} \text{ plus a generic point}\}$$

REMARK: asking also *finite presentations* we get $\mathrm{Spec}^{\mathtt{NC}}(\mathbb{Z}) = \mathrm{Spec}_G(\mathbb{Z})$
for it is not'suitable' for general noncommutative rings

# COMMUTATIVE EXAMPLES II

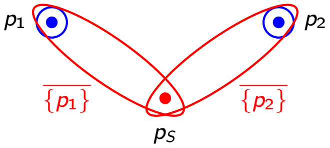
- For $\mathbb{K} \oplus \mathbb{K}$ , the localizations are two $\pi_i : \mathbb{K} \oplus \mathbb{K} \to \mathbb{K}$

*How to check if $\{\pi_1, \pi_2\}$ is a cover for $\mathbb{K} \oplus \mathbb{K}$?*

**PROPOSITION:** $\{A \to B_i\}_i$ is conservative if and only if for every $C \in \mathrm{HRings}_A$

$$C \cong 0 \iff C *^{\mathbb{L}}_A B_i \cong 0 \qquad \text{for all } i.$$

- $\{\pi_1, \pi_2\}$ is not conservative: $\mathrm{Mat}(2, \mathbb{K}) *^{\mathbb{L}}_{\mathbb{K} \oplus \mathbb{K}} \mathbb{K} \simeq 0$
- the noncommutative spectrum is $\mathrm{Spec}^{\mathtt{NC}}(\mathbb{K} \oplus \mathbb{K}) = \{p_1, p_2, p_s\}$

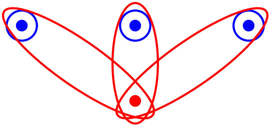


# NONCOMMUTATIVE EXAMPLE: path algebra of $A_2$ quiver

- For the path algebra $R = \mathbb{K}A_2$ over $\mathbb{K}$ of the $A_2$ quiver, the localizations are

$$\begin{tikzcd}
 & \mathbb{K}A_2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
P_1 \arrow[dr, dash] & P_2 \arrow[d, dash] & S_2 \arrow[dl, dash] \\
 & 0 &
\end{tikzcd}$$

- We have only trivial covers: $\{\mathrm{Id}_{\mathbf{0}}\}$, $\{\mathrm{Id}_{\boldsymbol{P_1}}\}$, $\{\mathrm{Id}_{\boldsymbol{P_2}}\}$, $\{\mathrm{Id}_{\boldsymbol{S_2}}\}$, and $\{\mathrm{Id}_{\boldsymbol{kA_2}}\}$
- $\mathrm{Spec}^{\mathrm{NC}}(\mathbb{K}A_2)$ is the spectral space



## THE FINE NONCOMMUTATIVE SPECTRUM

- Often we get trivial covers $\Longrightarrow$ many specialization points
- Checking if a family of localization forms a cover, it is not easy
- Not always possible to relate our spectrum with the Grothendieck one

**DEFINITION:** We call **fine formal homotopical Zariski topology**

- *open localization:* $A \to B$ homotopical epimorphism, i.e. $B *_A^{\mathbb{L}} B \simeq B$
- *fine covers* : $\{A \to B_i\}$ finite family of open localization s.t.

$$B_j = 0 \quad \Longleftrightarrow \quad B_j *_A^{\mathbb{L}} B_i \neq 0 \qquad \text{for all } i.$$

**PROPOSITION:** *fine* spectrum is *not functorial*, but there are canonical maps

$$\mathrm{Spec}^{\mathrm{NC}}(A) \longleftarrow \mathrm{Spec}^{\mathrm{NC}}_{\mathrm{fine}}(A) \longrightarrow \mathrm{Spec}_G(A)$$

- If $A \cong A_1 \times A_2 \Longrightarrow \mathrm{Spec}^{\mathrm{NC}}_{\mathrm{fine}}(A) \cong \mathrm{Spec}^{\mathrm{NC}}_{\mathrm{fine}}(A_1) \coprod \mathrm{Spec}^{\mathrm{NC}}_{\mathrm{fine}}(A_2)$ as Top
- If for $A, B$ are derived Morita equivalent $\Longrightarrow \mathrm{Spec}^{\mathrm{NC}}_{\mathrm{fine}}(A) \cong \mathrm{Spec}^{\mathrm{NC}}_{\mathrm{fine}}(B)$

## THE NONCOMMUTATIVE PRESHEAF

▶ For different rings, we might have the same $\mathrm{Spec}^{\mathtt{NC}} \Rightarrow$ we need a sheaf

MAIN ISSUE: Given $A \to B$ locazation,

- the natural structure pre-sheaf is not a sheaf on $\mathrm{Spec}^{\mathtt{NC}}_{\mathrm{fine}}$
- $(-) *^{\mathbb{L}}_A B$ and $(-) \otimes^{\mathbb{L}}_A B$ are not the same

DEFINITION: We call **noncommutative pre-sheaf** $\mathcal{O}_X : \mathrm{Ouv}(X)^{\mathrm{op}} \to \mathrm{HRings}_{\mathbb{Z}}$

$$\mathcal{O}_X(U) = \mathbb{R}\lim N^{\bullet}\left(\mathcal{O}_X(U_i)\right)$$

where $N^{\bullet}(\mathcal{O}_X(U_i))$ denotes the Čech nerve of the cover

THEOREM

- The following controvariant functor is faithful

$$\overline{\mathrm{Spec}^{\mathtt{NC}}} : \mathrm{HRings} \to \mathbf{PreRingSites} \qquad A \mapsto \left(\mathrm{Spec}^{\mathtt{NC}}(A), \mathcal{O}_A\right)$$

- **Rings** is equivalent to a subcategory of the category **PreRingSites**.

## FUTURE OUTLOOK

- To include $C^*$-algebras, we need a good raplacement of HRings:
  - $\mathrm{Hilb}_{\mathbb{C}}$ is quasi-abelian category, but not complete nor cocomplete
  - To remedy we consider $\mathrm{Ind}(\mathrm{Hilb}_{\mathbb{C}})$ and we define

$$\mathsf{A}\mathit{lg}\left(\mathsf{C}\mathit{h}^{\leq 0}(\mathrm{Ind}(\mathrm{Hilb}_{\mathbb{C}})), \widehat{\otimes}_{\mathbb{C}}\right)$$

  - Following Schneiders, we can identify a class of quasi-isomorphism $[W]$

$$\mathrm{HARing}_{\mathbb{C}} := \mathsf{Alg}\big(\mathsf{Ch}^{\leq 0}(\mathrm{Ind}(\mathrm{Hilb}_{\mathbb{C}})), \widehat{\otimes}_{\mathbb{C}}\big)[\mathrm{W}^{-1}]$$

- We can construct a spectrum using the formal homotopical Zarisky topology and using the fine topology, we shall obtain again a map to Gelfand's spectrum

*Open questions:*

- What is the noncommutative spectrum of the Weyl algebra?
- Is there a relation between irreducible representations and points of $\mathrm{Spec}^{\mathtt{NC}}$?

THANKS for your attention!